

# Scattering and self-adjoint extensions of the Aharonov-Bohm hamiltonian

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## Abstract

We consider the hamiltonian operator associated with planar sections of infinitely long cylindrical solenoids and with a homogeneous magnetic field in their interior. First, in the Sobolev space  $\mathcal{H}^2$ , we characterize all generalized boundary conditions on the solenoid border compatible with quantum mechanics, i.e., the boundary conditions so that the corresponding hamiltonian operators are self-adjoint. Then we study and compare the scattering of the most usual boundary conditions, that is, Dirichlet, Neumann and Robin.

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## 1 Introduction

Although the Aharonov-Bohm (AB) effect is a fundamental question in quantum physics, and despite the original work on the AB effect has been published 50 years ago [4, 18], it is still a very active area of research with many open mathematical questions. Here we address some of these questions and, our first aim is to try to characterize, in the two-dimensional space, all boundary conditions on the (cylindrical) solenoid border  $\mathcal{S}$  that are compatible with quantum mechanics, and whose domains are subspaces of the natural Sobolev space  $\mathcal{H}^2(\mathcal{S}')$ , where  $\mathcal{S}'$  is the exterior region of the solenoid.

Our (standard) cylindrical solenoid  $\mathcal{S}$  has radius  $a > 0$ , is infinitely long and centered at the origin (with axis coinciding with the  $z$  direction), it carries a stationary electric current so that there is a homogeneous magnetic field  $\mathbf{B} = (0, 0, B)$  confined to the solenoid interior  $\mathcal{S}^\circ$ , and vanishing in the exterior region  $\mathcal{S}'$ . A spinless charged particle of mass  $m = 1/2$  lives in  $\mathcal{S}'$  and has no contact with the magnetic field  $\mathbf{B}$ . If  $\mathbf{A}$  is a vector potential that generates such magnetic field, that is,  $\mathbf{B} = \nabla \times \mathbf{A}$ , the initial (quantum) AB hamiltonian operator for such charged particle is given by (with  $\hbar = 1$ ,  $c$  and  $q$  stand for the speed of light and particle electric charge, respectively)

$$H = \left( \mathbf{p} - \frac{q}{c} \mathbf{A} \right)^2, \quad \mathbf{p} = -i\nabla, \quad \text{dom } H = C_0^\infty(\mathcal{S}'). \quad (1)$$

Ahead, we will fix a specific choice of the vector potential.

This operator  $H$  is not self-adjoint and so does not correspond to a physical observable; the possible self-adjoint extensions characterize all possible physical interaction of the particle with the solenoid border (sometimes obtained through non-trivial limit procedures). It is important to note that the elements  $\psi$  of the domain  $C_0^\infty(\mathcal{S}')$  do not touch  $\mathcal{S}$  (in the sense that  $\psi = 0$  in a neighborhood of the solenoid) and  $H\psi$  has a very simple action. Such collection of “exotic” extensions might, for instance, include the description

of limit situations where singular perturbations (that could also depend on time) are slowly turned off.

It is usually assumed that the domain of the “physical” self-adjoint extensions is Dirichlet (i.e.,  $\psi = 0$  on  $\mathcal{S}$ ), and some theoretical arguments have appeared to justify such choice (see [9] and references therein). However, here we take an open-minded position and ask about other possibilities of boundary conditions. In a general sense, these other conditions correspond to the requirement of vanishing of the probability current at the solenoid border, and they may model different sorts of interactions between the particle and the solenoid. In Section 2 we characterize all of such boundary conditions whose domain of the corresponding self-adjoint hamiltonians are composed of functions with square integrable first and second derivatives (these are rather natural technical conditions in quantum mechanics). The operator  $H$  above has deficiency indices equal to infinity, and its self-adjoint extensions should be compared with the case of solenoid of zero radius discussed, for instance, in [2, 7], where the deficiency indices are equal to 2.

In Section 3 the two-dimensional scattering will be discussed in this context, but restricted to the traditional boundary conditions: Dirichlet, Neumann and Robin. In fact, the case of Dirichlet was investigated in [20] and part of our results are based on the techniques discussed therein. The scattering in case of solenoids with zero radius is discussed in [4, 2, 7, 14, 17].

For solenoids of radius greater than zero, and the above mentioned traditional extensions, we will show that the wave operators exist and are complete; we also find expressions for the scattering operators and their asymptotic behaviours for low and high energies. Finally, we will find explicitly the respective differential scattering cross sections (an important ingredient in experiments) and some figures will compare their values. From some point of view, the discrepancy among such figures could, in principle, be useful for an experimental selection of the boundary condition occurring in each situation; in principle it is not obvious which boundary conditions are naturally realized in laboratories, and we have found that given two of such self-adjoint extensions, it is always possible to find a range of energy so that the corresponding scattering cross sections can be distinguished.

## 2 Self-adjoint extensions

In this section we find and characterize an important class of self-adjoint extensions of the initial hamiltonian  $H$  (see equation (1)), with vector potential  $\mathbf{A}$  given, in polar coordinates  $(r, \theta)$ , by  $\mathbf{A} = (A_r, A_\theta)$ ,  $A_r \equiv 0$  and  $A_\theta = \frac{\Phi}{2\pi r}$ ,  $r \geq a$ , and  $\Phi$  is the total magnetic flux through the solenoid. As will be discussed in Section 3, this hermitian operator has deficiency indices [8]  $n_+(H) = n_-(H) = +\infty$ , and so it has infinitely many self-adjoint extensions.

The first physical and mathematical point to be addressed is to find some self-adjoint operators that may potentially describe the Aharonov-Bohm hamiltonian of a charged particle moving in the exterior region  $\mathcal{S}'$ . This particle can not penetrate the solenoid but interacts with  $\mathcal{S}$ , and the boundary conditions that give rise to self-adjoint realizations are the possible conditions, from the point of view of quantum mechanics, that may describe such interaction with different types of interface materials and limit procedures.

We note that in order to classify all such extensions it is necessary to make use of some Sobolev spaces  $\mathcal{H}^s(\mathcal{S})$  with  $s < 0$  [3]. Furthermore, another difficulty is that the domain of the adjoint operator

$$\text{dom } H^* = \{\psi \in L^2(\mathcal{S}') : H\psi \in L^2(\mathcal{S}')\} \quad (2)$$

is not contained in the space  $\mathcal{H}^2(\mathcal{S}')$  [11, 12, 13], which has proved to be natural in quantum problems. Below we shall restrict our arguments to the extensions whose domains are contained in  $\mathcal{H}^2(\mathcal{S}')$ , which will permit us to use boundary triples to find these extensions in a quite simpler way (see Remark 2).

Due to the symmetry of the problem, we shall consider a planar cross section; another fact is that the solenoid border in  $\mathbb{R}^3$  is not a compact set, and so it is not clear how to define the trace operators in the spatial case (and we want to avoid this technical point).

The reader interested only in the final results may go straightly to Theorem 2 and the examples that follow this theorem.

## 2.1 Boundary triples

Our way to find self-adjoint extensions of  $H$  is via boundary triples  $(\mathbf{h}, \rho_1, \rho_2)$ , as described in [8], Chapter 7, so we present a brief account of this technique.

**Definition 1.** Let  $T$  be a hermitian operator in a Hilbert space  $\mathcal{H}$ . The *boundary form* of  $T$  is the sesquilinear mapping  $\Gamma = \Gamma_{T^*} : \text{dom } T^* \times \text{dom } T^* \rightarrow \mathbb{C}$  given by

$$\Gamma(\xi, \eta) := \langle T^* \xi, \eta \rangle - \langle \xi, T^* \eta \rangle, \quad \xi, \eta \in \text{dom } T^*. \quad (3)$$

**Proposition 1.**  $\Gamma(\xi, \eta) = 0$ , for all  $\xi, \eta \in \text{dom } T^*$ , if, and only if,  $T^*$  is self-adjoint, that is, if, and only if,  $T$  is essentially self-adjoint.

Boundary forms can be used to determine self-adjoint extensions of  $T$  by noting that such extensions are restrictions of  $T^*$  to certain domains  $\mathcal{D}$  such that  $\Gamma(\xi, \eta) = 0$ , for all  $\xi, \eta \in \mathcal{D}$ . By von Neumann theory [8], each self-adjoint extension of  $T$  is in a one-to-one correspondence with unitary operators  $\hat{U} : K_-(T) \rightarrow K_+(T)$  between the deficiency subspaces  $K_\pm$  of  $T$ ;

denote by  $T^{\hat{U}}$  the corresponding self-adjoint extension whose domains is  $(\overline{T})$  denotes the closure of the operator  $T$ )

$$\text{dom } T^{\hat{U}} = \{\eta = \zeta + \eta_- - \hat{U}\eta_- : \zeta \in \text{dom } \overline{T}, \eta_- \in K_-(T)\}. \quad (4)$$

Note that the boundary form  $\Gamma$  restricted to  $\text{dom } T^{\hat{U}}$  vanishes.

Now we recall the concept of boundary triples.

**Definition 2.** Let  $T$  be a hermitian operator with deficiency indices  $n_-(T) = n_+(T)$ . A *boundary triple*  $(\mathbf{h}, \rho_1, \rho_2)$  for  $T$  is composed of a Hilbert space  $\mathbf{h}$  and two linear mappings  $\rho_1, \rho_2 : \text{dom } T^* \rightarrow \mathbf{h}$  with dense images and so that

$$b\Gamma_{T^*}(\xi, \eta) = \langle \rho_1(\xi), \rho_1(\eta) \rangle - \langle \rho_2(\xi), \rho_2(\eta) \rangle, \quad \forall \xi, \eta \in \text{dom } T^*, \quad (5)$$

for some constant  $0 \neq b \in \mathbb{C}$ .  $\langle \cdot, \cdot \rangle$  denotes the inner product in  $\mathbf{h}$  and  $\dim \mathbf{h} = n_+(T)$ .

Again, self-adjoint extensions of  $T$  are restrictions of  $T^*$  to certain domains  $\mathcal{D}$  so that  $\Gamma(\xi, \eta) = 0$ , for all  $\xi, \eta \in \mathcal{D}$ , and given a boundary triple for  $T$ , such domains  $\mathcal{D}$  are related to unitary operators  $U : \mathbf{h} \rightarrow \mathbf{h}$  so that  $U\rho_1(\xi) = \rho_2(\xi)$  and

$$\langle \rho_1(\xi), \rho_1(\eta) \rangle = \langle \rho_2(\xi), \rho_2(\eta) \rangle = \langle U\rho_1(\xi), U\rho_1(\eta) \rangle, \quad \forall \xi, \eta \in \mathcal{D}. \quad (6)$$

The main results we need here are summarized in the following theorem.

**Theorem 1.** Let  $T$  be a hermitian operator with equal deficiency indices. If  $(\mathbf{h}, \rho_1, \rho_2)$  is a boundary triple for  $T$ , then the self-adjoint extensions of  $T$  are given by

$$\begin{aligned} \text{dom } T^U &= \{\xi \in \text{dom } T^* : \rho_2(\xi) = U\rho_1(\xi)\}, \\ T^U \xi &= T^* \xi, \quad \xi \in \text{dom } T^U, \end{aligned} \quad (7)$$

for each unitary operator  $U : \mathbf{h} \rightarrow \mathbf{h}$ .

## 2.2 Boundary triples for the AB operator

Some self-adjoint extensions of the initial AB operator  $H$  will be found. It will combine the cylindrical symmetry with the topological property of multiply connectedness, that is, the plane with a circular hole, without mentioning the important ingredient of a magnetic potential  $\mathbf{A}$  with  $\text{div } \mathbf{A} = 0$  in  $\mathcal{S}'$ . The method can be adapted to other regions with boundaries so that the trace construction applies (e.g., smooth and compact boundaries).

Although a  $\psi(r, \theta) \in \mathcal{H}^1(\mathcal{S}')$  is not necessarily continuous, it is possible to give a meaning to the restriction  $\psi(a, \theta) = \psi|_{\mathcal{S}}(\theta) \in L^2(\mathcal{S})$  via the so-called *trace* (more properly, it should be called *Sobolev trace*) of  $\psi$ ; see ahead. It

turns out that there is a continuous linear mapping  $\gamma : C_0^1(\mathbb{R}^2) \subset \mathcal{H}^1(\mathcal{S}') \rightarrow L^2(\mathcal{S})$ ,  $\gamma(\varphi(r, \theta)) = \varphi(a, \theta)$ , that is, there is  $C > 0$  so that

$$\|\gamma\varphi\|_{L^2(\mathcal{S})} = \|\varphi(a, \theta)\|_{L^2(\mathcal{S})} \leq C \|\varphi\|_{\mathcal{H}^1(\mathcal{S}')}, \quad \varphi \in C_0^1(\mathbb{R}^2). \quad (8)$$

Note that for  $\varphi \in C_0^1(\mathbb{R}^2)$  the boundary values  $\varphi(a, \theta)$  are well defined for any angle  $\theta$ . By density, this mapping has a unique continuous extension  $\gamma_0 : \mathcal{H}^1(\mathcal{S}') \rightarrow L^2(\mathcal{S})$ , called the *trace mapping* (see chapters 1 and 2 of [15] and also [3, 6]), and one defines  $\psi(a, \theta) := (\gamma_0\psi)(\theta)$  for all  $\psi \in \mathcal{H}^1(\mathcal{S}')$ .

Similarly it is defined the trace mapping

$$\gamma_1 : \mathcal{H}^2(\mathcal{S}') \rightarrow L^2(\mathcal{S}), \quad \frac{\partial\psi}{\partial\vec{n}} \Big|_{\mathcal{S}} = \gamma_1\psi, \quad (9)$$

where  $\vec{n}$  is the normalized vector normal to  $\mathcal{S}$  pointing to inside the solenoid.

We shall also make use of the Green's formulae

$$\int_{\mathcal{S}'} \Delta\psi(x, y)\varphi(x, y) dx dy + \int_{\mathcal{S}'} \nabla\psi(x, y)\nabla\varphi(x, y) dx dy = \int_{\mathcal{S}} \gamma_1\psi \gamma_0\varphi d\sigma, \quad (10)$$

which holds for all  $\psi, \varphi \in \mathcal{H}^2(\mathcal{S}')$ , and

$$\int_{\mathcal{S}'} \frac{\partial\psi}{\partial x}(x, y)\varphi(x, y) dx dy + \int_{\mathcal{S}'} \psi(x, y)\frac{\partial\varphi}{\partial x}(x, y) dx dy = \int_{\mathcal{S}} \gamma_0\psi \gamma_0\varphi \gamma_0(\vec{n} \cdot \vec{e}_x) d\sigma, \quad (11)$$

$$\int_{\mathcal{S}'} \frac{\partial\psi}{\partial y}(x, y)\varphi(x, y) dx dy + \int_{\mathcal{S}'} \psi(x, y)\frac{\partial\varphi}{\partial y}(x, y) dx dy = \int_{\mathcal{S}} \gamma_0\psi \gamma_0\varphi \gamma_0(\vec{n} \cdot \vec{e}_y) d\sigma, \quad (12)$$

which hold for all  $\psi, \varphi \in \mathcal{H}^1(\mathcal{S}')$ , where  $d\sigma$  is the “surface” measure in  $\mathcal{S}$  (recall that here  $\mathcal{S}$  is the circle centered at the origin and radius  $a > 0$ ),  $(x, y) \in \mathbb{R}^2$  and  $\vec{e}_x, \vec{e}_y$  are the unit vectors along the axes  $x$  and  $y$ , respectively.

Note that the kernel of the trace operator  $\gamma_0$  is the Hilbert space

$$\mathcal{H}_0^1(\mathcal{S}') := \{\psi \in \mathcal{H}^1(\mathcal{S}') : (\gamma_0\psi)(\theta) = \psi(a, \theta) = 0\}, \quad (13)$$

which can also be defined as the closure of  $C_0^\infty(\mathcal{S}')$  in  $\mathcal{H}^1(\mathcal{S}')$ .

Now we introduce a boundary form  $\Gamma$  for the initial AB operator (1), and companion mappings  $\rho_1$  and  $\rho_2$  as well. The boundary form of  $H$ , for  $\psi, \varphi \in \text{dom } H^*$ , is

$$\Gamma(\psi, \varphi) := \langle H^*\psi, \varphi \rangle - \langle \psi, H^*\varphi \rangle, \quad (14)$$

and by restricting to those self-adjoint extensions whose domains are contained in  $\mathcal{H}^2(\mathcal{S}')$ , Sobolev traces can be invoked. Since  $\text{div } \mathbf{A} = 0$ , the boundary form of  $H$  is found to be given by

$$\begin{aligned} \Gamma(\psi, \varphi) &= \langle H^*\psi, \varphi \rangle - \langle \psi, H^*\varphi \rangle \\ &= \int_{\mathcal{S}} (\overline{\gamma_0\psi} \gamma_1\varphi - \overline{\gamma_1\psi} \gamma_0\varphi) d\sigma - 2i \int_{\mathcal{S}} (\overline{\gamma_0\psi} \gamma_0(\mathbf{A} \cdot \vec{n}) \gamma_0\varphi) d\sigma, \end{aligned} \quad (15)$$

for all  $\psi, \varphi \in \mathcal{H}^2(\mathcal{S}')$ .

By passing to polar coordinates  $(r, \theta)$ , the above boundary form  $\Gamma$  may be rewritten as

$$\Gamma(\psi, \varphi) = a \int_0^{2\pi} \left( \overline{\psi(a, \theta)} \frac{\partial \varphi}{\partial r}(a, \theta) - \overline{\frac{\partial \psi}{\partial r}(a, \theta)} \varphi(a, \theta) - 2i \overline{\psi(a, \theta)} (\mathbf{A} \cdot \vec{r})(a, \theta) \varphi(a, \theta) \right) d\theta, \quad (16)$$

for all  $\psi, \varphi \in \mathcal{H}^2(\mathcal{S}')$ , with  $\chi(a, \theta)$  and  $\frac{\partial \chi}{\partial r}(a, \theta)$  denoting the traces  $\gamma_0 \chi$  and  $\gamma_1 \chi$ , respectively, for all  $\chi \in \mathcal{H}^2(\mathcal{S}')$ .

Now we introduce the boundary triple  $(\mathbf{h}, \rho_1, \rho_2)$ , with  $\mathbf{h} = \mathbf{L}^2(\mathcal{S})$ , acting in  $\mathcal{H}^2(\mathcal{S}')$  by  $\rho_j : \mathcal{H}^2(\mathcal{S}') \rightarrow \mathbf{L}^2(\mathcal{S})$ ,  $j = 1, 2$ ,

$$\begin{aligned} \rho_1(\psi) &= \psi(a, \theta) + i \left( \frac{\partial \psi}{\partial r}(a, \theta) - i(\mathbf{A} \cdot \vec{r})(a, \theta) \psi(a, \theta) \right), \\ \rho_2(\psi) &= \psi(a, \theta) - i \left( \frac{\partial \psi}{\partial r}(a, \theta) - i(\mathbf{A} \cdot \vec{r})(a, \theta) \psi(a, \theta) \right). \end{aligned} \quad (17)$$

After a short calculation it follows that

$$(2i/a) \Gamma(\psi, \varphi) = \langle \rho_1(\psi), \rho_1(\varphi) \rangle_{\mathbf{L}^2(\mathcal{S})} - \langle \rho_2(\psi), \rho_2(\varphi) \rangle_{\mathbf{L}^2(\mathcal{S})}, \quad (18)$$

for all  $\psi, \varphi \in \mathcal{H}^2(\mathcal{S}')$ .

Since  $\mathbf{A} \cdot \vec{r} = 0$ , the expressions of  $\rho_1$  and  $\rho_2$  are reduced to

$$\begin{aligned} \rho_1(\psi) &= \psi(a, \theta) + i \frac{\partial \psi}{\partial r}(a, \theta), \\ \rho_2(\psi) &= \psi(a, \theta) - i \frac{\partial \psi}{\partial r}(a, \theta), \end{aligned} \quad (19)$$

and the vector potential no longer appears in these expressions; note that this was possible only due to the cylindrical symmetry of the problem.

Finally, by applying Theorem 1, the above constructions permit us to conclude the main result of this section:

**Theorem 2.** *All self-adjoint extensions  $H^U$  of  $H$ , acting in  $\mathcal{H}^2(\mathcal{S}')$ , are characterized by unitary operators  $U : \mathbf{L}^2(\mathcal{S}) \rightarrow \mathbf{L}^2(\mathcal{S})$  so that  $\rho_2(\psi) = U \rho_1(\psi)$ , that is,*

$$\begin{aligned} \text{dom } H^U &= \left\{ \psi \in \mathcal{H}^2(\mathcal{S}') : (\mathbf{1} - U) \psi(a, \theta) = i(\mathbf{1} + U) \frac{\partial \psi}{\partial r}(a, \theta) \right\}, \\ H^U \psi &= H^* \psi, \quad \psi \in \text{dom } H^U. \end{aligned} \quad (20)$$

### 2.3 Some self-adjoint extensions of $H$

For sake of completeness, in what follows we present some particular choices of unitary operators  $U$  and the corresponding self-adjoint extensions [8] of the initial AB operator  $H$ .

**Example 1.** If  $U = -\mathbf{1}$ , then

$$\begin{aligned} \text{dom } H^U &= \{\psi \in \mathcal{H}^2(\mathcal{S}') : \psi(a, \theta) = 0\} = \mathcal{H}^2(\mathcal{S}') \cap \mathcal{H}_0^1(\mathcal{S}'), \\ H^U \psi &= H^* \psi, \quad \psi \in \text{dom } H^U. \end{aligned} \quad (21)$$

This is the so-called Dirichlet self-adjoint realization, which is usually assumed to be the physical relevant in the literature [20, 9].

**Example 2.** If  $U = \mathbf{1}$ , then

$$\begin{aligned} \text{dom } H^U &= \{\psi \in \mathcal{H}^2(\mathcal{S}') : \partial\psi/\partial r(a, \theta) = 0\}, \\ H^U \psi &= H^* \psi, \quad \psi \in \text{dom } H^U. \end{aligned} \quad (22)$$

This is the so-called Neumann self-adjoint realization.

**Example 3.** Assume that  $(\mathbf{1} + U)$  is invertible.

In this case, to each self-adjoint operator  $A : \text{dom } A \subset L^2(\mathcal{S}) \rightarrow L^2(\mathcal{S})$  corresponds a self-adjoint extension  $H^A$ . In fact, first pick a unitary operator  $U_A$  so that  $A = -i(\mathbf{1} - U_A)(\mathbf{1} + U_A)^{-1}$ ,  $\text{dom } A = \text{rng } (\mathbf{1} + U_A)$  and  $\text{rng } A = \text{rng } (\mathbf{1} - U_A)$ ; remind of Cayley transform. Now,  $\text{dom } H^A$  is the set of  $\psi \in \mathcal{H}^2(\mathcal{S}')$  with “ $\partial\psi/\partial r(a, \cdot) = A\psi(a, \cdot)$ ,” understood in the sense that

$$(\mathbf{1} - U_A) \psi(a, \theta) = i(\mathbf{1} + U_A) \frac{\partial\psi}{\partial r}(a, \theta), \quad (23)$$

in order to avoid domain questions. Of course the quotation marks may be removed in case the operator  $A$  is bounded.

Similarly, for each self-adjoint  $B$  acting in  $L^2(\mathcal{S})$  there corresponds a unitary  $U_B$ , and if  $(\mathbf{1} - U_B)$  is invertible, then it corresponds the self-adjoint extension  $H^B$  of  $H$  with  $\text{dom } H^B$  being the set of  $\psi \in \mathcal{H}^2(\mathcal{S}')$  so that “ $\psi(a, \cdot) = B \frac{\partial\psi}{\partial r}(a, \cdot)$ ,” in the sense that

$$(\mathbf{1} - U_B) \psi(a, \theta) = i(\mathbf{1} + U_B) \frac{\partial\psi}{\partial r}(a, \theta). \quad (24)$$

The quotation marks may be removed in case the operator  $B$  is bounded.

Note that Example 4 ahead is, in fact, particular cases of this example in which  $A = \mathcal{M}_f$  and  $B = \mathcal{M}_g$  are multiplication operators.

**Example 4.**  $U$  is a multiplication operator.



Given a real-valued (measurable) function  $u(\theta)$  defined on  $\mathcal{S}$ , put  $U = \mathcal{M}_{e^{iu(\theta)}}$ . If the set  $\{\theta : \exp(iu(\theta)) = -1\}$  has measure zero, then the function

$$f(\theta) = -i \frac{1 - e^{iu(\theta)}}{1 + e^{iu(\theta)}} \quad (25)$$

is (measurable) well defined and real valued. The domain of the corresponding self-adjoint extension  $H^U$  of  $H$  is

$$\text{dom } H^U = \{\psi \in \mathcal{H}^2(\mathcal{S}') : \partial\psi/\partial r(a, \theta) = f(\theta)\psi(a, \theta)\}. \quad (26)$$

Similarly, if  $\{\theta : \exp(iu(\theta)) = 1\}$  has measure zero,

$$g(\theta) = i \frac{1 + e^{iu(\theta)}}{1 - e^{iu(\theta)}} \quad (27)$$

is real valued and the domain of the subsequent self-adjoint extension  $H^U$  of  $H$  is

$$\text{dom } H^U = \{\psi \in \mathcal{H}^2(\mathcal{S}') : \psi(a, \theta) = g(\theta)\partial\psi/\partial r(a, \theta)\}. \quad (28)$$

Special cases are given by constant functions  $f, g$ , the so-called Robin self-adjoint realization. We shall discuss the scattering for this extension in Section 3.

**Remark 1.** *Since the deficiency indices of  $H$  are infinite, there is a plethora of self-adjoint extensions of  $H$  in the multiply connected domain  $\mathcal{S}'$ . Some of them can be quite unusual and hard to understand from the physical and mathematical points of view.*

**Remark 2.** *By using a continuous extension of the trace maps to the dual Sobolev spaces  $\mathcal{H}^{-1/2}(\mathcal{S})$  and  $\mathcal{H}^{-3/2}(\mathcal{S})$ , in [12] one finds references and comments to her previous works on all self-adjoint extensions of the laplacian in terms of self-adjoint operators from closed subspaces of  $\mathcal{H}^{-1/2}(\mathcal{S})$ . It is possible to follow those works and apply the same technique to find all self-adjoint extensions of the initial AB operator  $H$ , but we will not describe them here since the characterizations are rather abstract, involve spaces not usual in quantum mechanics, and they require a lengthy construction that is not so clean as the extensions we have found in Theorem 2 (which also includes the traditional self-adjoint extensions we are most interested in).*

### 3 Scattering

In this section we study and compare the scattering for the Robin self-adjoint realizations (which includes Neumann and Dirichlet as particular cases) of the initial AB operator  $H$ .

### 3.1 Scattering theory: a brief account

Now we briefly recall results from scattering theory, based mainly on [20, 5], focused on what we want to do in the next sections. For details and a thorough study of the mathematics of scattering theory see [19, 21].

Consider a system in (non-relativistic) quantum mechanics whose states  $\xi$  are unit vectors in a Hilbert space  $\mathcal{H}$  and whose time evolution is generated by a self-adjoint operator  $h$  acting in  $\mathcal{H}$ , and let  $H_0$  denote the free hamiltonian acting in  $\mathcal{H}_0$ . The question is whether the states  $e^{-iht}\xi$  are scattering states, i.e., if there are free states  $\xi_{\pm} \in \mathcal{H}_0$  so that

$$\left\| e^{-iht}\xi - \mathcal{J}e^{-iH_0t}\xi_{\pm} \right\| = \left\| \xi - e^{iht}\mathcal{J}e^{-iH_0t}\xi_{\pm} \right\| \quad (29)$$

vanishes as  $t \rightarrow \pm\infty$ . A comparison mapping  $\mathcal{J} : \mathcal{H}_0 \rightarrow \mathcal{H}$ , which is a unitary operator (or just a bounded one), is sometimes conveniently introduced.

**Definition 3.** The *wave operators* are the strong limits

$$\mathcal{W}_{\pm} := \text{s-}\lim_{t \rightarrow \pm\infty} e^{iht}\mathcal{J}e^{-iH_0t}, \quad (30)$$

if they exist.

Recall that the wave operators  $\mathcal{W}_{\pm}$  are said to be *complete* if  $\text{rng } \mathcal{W}_{\pm} = \mathcal{H}_p(h)^{\perp}$ , where  $\mathcal{H}_p(h)$  denotes the closure of the subspace spanned by the eigenvectors of  $h$ .

The vectors  $\xi_{\pm}$  and  $\xi$  satisfy the relation  $\xi = \mathcal{W}_{\pm}\xi_{\pm}$  and the wave operators  $\mathcal{W}_{\pm} : \text{dom } \mathcal{W}_{\pm} \rightarrow \text{rng } \mathcal{W}_{\pm}$  are partial isometries. Thus,  $\mathcal{W}_{\pm}\mathcal{W}_{\pm}^*$  are orthogonal projections onto  $\text{rng } \mathcal{W}_{\pm}$ , and restricted to these subspaces  $\mathcal{W}_{\pm}^* = \mathcal{W}_{\pm}^{-1}$ . Furthermore,

$$\xi_+ = S\xi_-, \quad (31)$$

where  $S := \mathcal{W}_+^*\mathcal{W}_-$  is the so-called *scattering operator* or *S-matrix*.

The physical system we consider is a scattering of particles off a cylindrical obstacle in the plane of points  $\vec{x} = (x, y)$ , and the Hilbert space is  $\mathcal{H} = L^2(\mathcal{S}')$ . Moreover, we are also interested in the scattering away from a short range continuous potential  $V(x, y)$ , which we also assume that it is spherically symmetric, that is,  $V(x, y) = V(r)$ ,  $r = |(x, y)|$ ; short range means that there are constants  $C, R > 0$  so that

$$|V(x, y)| \leq \frac{C}{r^{1+\delta}}, \quad \forall r > R, \quad (32)$$

for some  $\delta > 0$ . Let

$$h = -\Delta + V(r) \quad \text{and} \quad H_0 = p_1^2 + p_2^2, \quad (33)$$

acting in the position space  $\mathcal{H}$  and momentum  $\mathcal{H}_0 = L^2(\hat{\mathbb{R}}^2)$ , respectively, where  $\vec{p} = (p_1, p_2)$ ,  $p^2 = |\vec{p}|^2$  and the comparison operator  $\mathcal{J}$  is the inverse Fourier transform  $\mathcal{F}$ .

In the time-independent scattering theory one solves the time-independent Schrödinger equation  $h\varphi = p^2\varphi$  for the incoming  $\varphi_-(\vec{x}, \vec{p})$  and outgoing  $\varphi_+(\vec{x}, \vec{p})$  functions, which are reduced to a plane wave  $\phi(\vec{x}, \vec{p}) = e^{i\vec{x}\cdot\vec{p}}$  for  $|\vec{x}| \rightarrow \infty$  (these are solutions of the Schrödinger equation of the free particle). One has the following connection

$$(\mathcal{W}_\pm \zeta)(\vec{x}) = \frac{1}{2\pi} \int \varphi_\pm(\vec{x}, \vec{p}) \zeta(\vec{p}) d\vec{p}, \quad \zeta \in \mathcal{H}_0. \quad (34)$$

By employing polar coordinates, both in space position  $(r, \theta)$  and in the space of momenta  $(k, \theta')$ , one obtains

$$h = -\frac{\partial^2}{\partial r^2} - \frac{1}{r} \frac{\partial}{\partial r} - \frac{1}{r^2} \frac{\partial^2}{\partial \theta^2} + V(r) \quad \text{and} \quad H_0 = k^2, \quad (35)$$

respectively. One then considers the asymptotic behaviour of the solution (incoming wave function) of the time-independent Schrödinger equation

$$\varphi_-(r, \theta; k, \theta') \sim e^{ikr \cos(\theta - \theta')} + f(k, \theta - \theta') \frac{e^{ikr}}{r^{1/2}}, \quad r \rightarrow \infty, \quad (36)$$

where  $f$  is the *scattering amplitude*, and so the *differential scattering cross section* is

$$\left( \frac{d\sigma}{d\theta} \right) (k, \theta) = |f(k, \theta)|^2. \quad (37)$$

Physically, this quantity measures the probability density of an incident particle, after interaction with the target (scatterer center), i.e., a scattered particle to be found within of a cone around  $(k, \theta)$ .

Now we recall how to find the asymptotic behaviour of  $\varphi_-$  and the scattering amplitude. If  $J_n$  denotes the Bessel function of first kind of order  $n$ , one has

$$e^{ikr \cos \theta} = \sum_{m=-\infty}^{\infty} i^{|m|} J_{|m|}(kr) e^{im\theta}, \quad f(k, \theta) = \sum_{m=-\infty}^{\infty} f_m(k) e^{im\theta}, \quad (38)$$

and by (36), the asymptotic behaviour of  $J_n(r)$  [16],

$$J_n(r) \sim \left( \frac{2}{\pi r} \right)^{1/2} \cos \left( r - \frac{1}{2}n\pi - \frac{\pi}{4} \right), \quad r \rightarrow \infty, \quad (39)$$

and recalling that  $\cos \vartheta = (e^{i\vartheta} + e^{-i\vartheta})/2$ , one obtains

$$\varphi_-(r, \theta; k, 0) \sim \sum_{m=-\infty}^{\infty} \left[ \frac{(-1)^m e^{i\pi/4 - ikr}}{(2\pi kr)^{1/2}} + \left( \frac{e^{-i\pi/4}}{(2\pi kr)^{1/2}} + \frac{f_m}{r^{1/2}} \right) e^{ikr} \right] e^{im\theta}. \quad (40)$$

On the other hand,  $\varphi_-$  solves the Schrödinger equation and is regular at the origin. By using separation of variables,

$$\varphi_-(r, \theta; k, 0) = \sum_{m=-\infty}^{\infty} a_m(k) \varphi_m(r, k) e^{im\theta}, \quad (41)$$

where  $\varphi_m$  is a solution for radial Schrödinger equation of angular momentum  $m$ ,

$$\left( -\frac{d^2}{dr^2} - \frac{1}{r} \frac{d}{dr} + \frac{m^2}{r^2} + V(r) \right) \varphi = k^2 \varphi, \quad (42)$$

which is regular at the origin. Since  $V$  is short range, it follows that the behaviour of  $\varphi_m$  for  $r \rightarrow \infty$  is given by

$$\varphi_m(r, k) \sim \left( \frac{2}{\pi k r} \right)^{1/2} \cos \left( kr - \frac{1}{2} |m| \pi - \frac{\pi}{4} + \delta_m(k) \right), \quad (43)$$

and herein the phase shift  $\delta_m$  was introduced. The phase shift is a measure of the argument difference to the asymptotic behaviour of the solution  $J_{|m|}(kr)$  to the radial free equation that is regular at the origin.

The comparison of the asymptotic behaviour of  $\varphi_-$  given by (40), with the one given by (41) and (43), gives us the important relations

$$a_m(k) = i^{|m|} e^{i\delta_m(k)}, \quad (44)$$

$$f_m(k) = \frac{e^{2i\delta_m(k)} - 1}{(2\pi i k)^{1/2}}, \quad (45)$$

and

$$f(k, \theta) = \frac{1}{(2\pi i k)^{1/2}} \sum_{m=-\infty}^{\infty} \left( e^{2i\delta_m(k)} - 1 \right) e^{im\theta}, \quad (46)$$

which formally expresses the scattering amplitude in terms of the phase shifts  $\delta_m$ . Later on, in our applications to the extensions of the AB hamiltonian, these relations will be considered from the point of view of distribution theory.

Now a bit of the time-dependent approach again. Decompose the Hilbert spaces  $\mathcal{H}$  and  $\mathcal{H}_0$  in subspaces  $\mathfrak{h}_m$ ,  $\mathfrak{h}_{0,m}$ , with corresponding projections  $P_m$ ,  $P_{0,m}$ , respectively. For example,

$$(P_m \eta)(r, \theta) = \frac{e^{im\theta}}{(2\pi)^{1/2}} \eta_m(r), \quad (47)$$

where

$$\eta_m(r) = \frac{1}{(2\pi)^{1/2}} \int_0^{2\pi} e^{-im\theta} \eta(r, \theta) d\theta \quad (48)$$

are the components of  $\eta$  with angular momentum  $m$ , belonging to the Hilbert space  $\mathcal{H}_r = L^2_{rdr}([0, \infty))$ . The subspaces  $\mathfrak{h}_m$  are invariant under the hamiltonian  $h$  and the subsequent restriction is given by

$$h_m = -\frac{d^2}{dr^2} - \frac{1}{r} \frac{d}{dr} + \frac{m^2}{r^2} + V(r) \quad (49)$$

in  $\mathcal{H}_r$ . For the inverse Fourier transform  $\mathcal{F}$  one has

$$(\mathcal{F}\zeta)(r, \theta) = (2\pi)^{-1/2} \sum_{m=-\infty}^{\infty} e^{im\theta} (\mathcal{F}_m \zeta_m)(r), \quad (50)$$

with  $\mathcal{F}_m : \mathcal{H}_k \rightarrow \mathcal{H}_r$  denoting the unitary operator

$$(\mathcal{F}_m \psi)(r) = i^{|m|} \int_0^\infty J_{|m|}(kr) \psi(k) k dk, \quad (51)$$

where  $\mathcal{H}_k = L^2_{kdk}([0, \infty))$ . Correspondingly, we consider a sequence of wave operators from  $\mathcal{H}_k$  into  $\mathcal{H}_r$

$$\mathcal{W}_{\pm, m} = \text{s-} \lim_{t \rightarrow \pm\infty} e^{ih_m t} \mathcal{F}_m e^{-ik^2 t}. \quad (52)$$

Since  $\mathcal{W}_{\pm, m}$  exist and are complete in each sector  $m$ , one has the following relation with the time-independent approach

$$(\mathcal{W}_{\pm, m} \psi)(r) = i^{|m|} \int_0^\infty \varphi_m(k, r) e^{\mp i\delta_m(k)} \psi(k) k dk, \quad (53)$$

so that the corresponding  $S$ -matrix in the sector  $m$ ,  $S_m : \mathcal{H}_k \rightarrow \mathcal{H}_k$ ,

$$S_m = \mathcal{W}_{+, m}^* \mathcal{W}_{-, m}, \quad (54)$$

is given, after some calculations, by

$$(S_m \psi)(k) = \begin{cases} e^{2i\delta_m(k)} \psi(k), & k = k' \\ 0, & k \neq k' \end{cases}, \quad (55)$$

that is,  $S_m$  is simply the multiplication operator by  $e^{2i\delta_m(k)}$  on  $\mathcal{H}_k$ .

The wave operator in Definition 3 can now be written as

$$(\mathcal{W}_{\pm} \zeta)(r, \theta) = (2\pi)^{-1/2} \sum_{m=-\infty}^{\infty} e^{im\theta} (\mathcal{W}_{\pm, m} \zeta_m)(r). \quad (56)$$

Similarly, the scattering operator  $S : \mathcal{H}_0 \rightarrow \mathcal{H}_0$  is found to satisfy

$$\begin{aligned} \langle \zeta, S\xi \rangle &= \langle \zeta, \xi \rangle + (2\pi)^{-1} \sum_{m=-\infty}^{\infty} \int_0^\infty \int_0^{2\pi} \int_0^{2\pi} e^{im(\theta-\theta')} \overline{\zeta(k, \theta)} \\ &\quad \times \xi(k, \theta') \left( e^{2i\delta_m(k)} - 1 \right) d\theta d\theta' k dk, \end{aligned} \quad (57)$$

and by equation (46), we conclude that

$$\begin{aligned} \langle \zeta, S\xi \rangle &= \langle \zeta, \xi \rangle + \int_0^\infty \int_0^{2\pi} \int_0^{2\pi} \overline{\zeta(k, \theta)} \xi(k, \theta') \\ &\quad \times \left( \frac{ik}{2\pi} \right)^{1/2} f(k, \theta - \theta') d\theta d\theta' k dk, \end{aligned} \quad (58)$$

for all  $\zeta, \xi \in \mathcal{H}_0$ , where  $\langle \cdot, \cdot \rangle$  denotes the inner product in  $\mathcal{H}_0$ , and so

$$“(S - \mathbf{1})(k, \theta) = \left( \frac{ik}{2\pi} \right)^{1/2} f(k, \theta),” \quad (59)$$

understood in the sense described above, which is the correct relation between  $f$  and  $S$ .

### 3.2 Robin self-adjoint extensions

In this subsection we describe the self-adjoint extensions of the initial AB operator (1) for which we will study scattering. We choose some of the self-adjoint extensions that preserve angular momentum since we are considering that the subspaces  $\mathfrak{h}_m$  are invariants under  $H$ . In addition, the extensions described below are the most common and studied in the literature when borders are considered. Note that there are extensions that do not preserve angular momentum.

In order to simplify expressions, we will take  $c = q = 1$  in our initial hermitian operator (1),

$$H = (-i\nabla - \mathbf{A})^2, \quad (60)$$

which acts in a subspace of the Hilbert space  $\mathcal{H} = L^2(\mathcal{S}')$ , and recall that the vector potential  $\mathbf{A}$ , in polar coordinates  $(r, \theta)$ , is given by  $\mathbf{A} = (A_r, A_\theta)$ , with  $A_r \equiv 0$  and  $A_\theta = \frac{\Phi}{2\pi r}$ ,  $r \geq a$ . This operator can be written in polar coordinates as

$$H = -\frac{\partial^2}{\partial r^2} - \frac{1}{r} \frac{\partial}{\partial r} + \frac{1}{r^2} \left( i \frac{\partial}{\partial \theta} - \alpha \right)^2, \quad (61)$$

where  $\alpha = -\Phi/(2\pi)$ . Without loss of generality, consider  $0 \leq \alpha < 1$ , and  $\alpha = 0$  means that no magnetic field is present.

The next step is to construct the self-adjoint extensions of  $H$  we are interested in. By making the polar decomposition  $\mathcal{H} = \mathcal{H}_r^a \otimes \mathcal{H}_\theta$ , where  $\mathcal{H}_r^a = L_{rdr}^2[a, \infty)$  and  $\mathcal{H}_\theta = L^2[0, 2\pi]$ , we obtain a sequence of formal restriction operators to  $\mathfrak{h}_m$

$$H_{m+\alpha} = -\frac{d^2}{dr^2} - \frac{1}{r} \frac{d}{dr} + \frac{(m+\alpha)^2}{r^2} \quad (62)$$

in  $\mathcal{H}_r^a$ . To achieve our goal we need to turn these operators into self-adjoint ones acting in  $\mathcal{H}_r^a$ . They are not essentially self-adjoint on  $C_0^\infty(a, \infty)$  for any

$m + \alpha$ . To see this, note that the potential term  $(m + \alpha)^2/r^2$  is bounded in  $\mathcal{H}_r^a$ , hence we need only to consider the differential operator

$$h_1 = -\frac{d^2}{dr^2} - \frac{1}{r} \frac{d}{dr} + \frac{1}{4r^2}. \quad (63)$$

But using the unitary operator  $U : \mathcal{H}_r^a \rightarrow L_{dr}^2[a, \infty)$ , given by  $(U\psi)(r) = r^{1/2}\psi(r)$ , the operator  $h_1$  becomes

$$h_2 = Uh_1U^{-1} = -\frac{d^2}{dr^2}, \quad (64)$$

which is not essentially self-adjoint on  $C_0^\infty(a, \infty)$  since the functions  $u_\pm(r) = e^{-e^{\pm i\pi/4}r} \in L_{dr}^2[a, \infty)$  and satisfy  $h_2^*u \pm iu = 0$ ; in other words, its deficiency indices are  $n_-(h_2) = n_+(h_2) = 1$ . Since this holds for all  $m$ , it justifies the assertion that  $n_\pm(H) = \infty$  in Remark 1.

However, we can find all self-adjoint extensions of  $h_2$  [8], which are well known and given by

$$\text{dom } h_2^{\tilde{\lambda}} = \left\{ \psi \in \mathcal{H}^2[a, \infty) : \psi(a) = \tilde{\lambda}\psi'(a) \right\}, \quad h_2^{\tilde{\lambda}}\psi = h_2\psi, \quad (65)$$

for each  $\tilde{\lambda} \in \mathbb{R} \cup \{\infty\}$ .

Thus, we have the corresponding self-adjoint extensions of  $H_{m+\alpha}$

$$\text{dom } H_{m+\alpha}^{\tilde{\lambda}} = U^{-1}(\text{dom } h_2^{\tilde{\lambda}}), \quad H_{m+\alpha}^{\tilde{\lambda}}\psi = H_{m+\alpha}\psi, \quad (66)$$

that is,

$$\begin{aligned} \text{dom } H_{m+\alpha}^{\tilde{\lambda}} &= \left\{ u = r^{-1/2}\psi : \psi \in \mathcal{H}^2[a, \infty) \text{ and } \psi(a) = \tilde{\lambda}\psi'(a) \right\} \\ &= \left\{ u \in U^{-1}(\mathcal{H}^2[a, \infty)) : r^{1/2}u(r)|_{r=a} = \tilde{\lambda} \frac{d}{dr}[r^{1/2}u(r)]|_{r=a} \right\}. \end{aligned} \quad (67)$$

Therefore, the boundary conditions that characterize the Robin self-adjoint extensions of  $H_{m+\alpha}$  are given by  $(2a - \tilde{\lambda})u(a) = 2a\tilde{\lambda}u'(a)$ . If  $\tilde{\lambda} \neq 2a$ , then

$$\text{dom } H_{m+\alpha}^{\tilde{\lambda}} = \left\{ u \in U^{-1}(\mathcal{H}^2[a, \infty)) : u(a) = \frac{2a\tilde{\lambda}}{2a - \tilde{\lambda}}u'(a) \right\}. \quad (68)$$

We shall denote these Robin self-adjoint extensions by  $H_{m+\alpha}^\lambda$ , that is

$$\begin{aligned} \text{dom } H_{m+\alpha}^\lambda &= \left\{ u \in U^{-1}(\mathcal{H}^2[a, \infty)) : u(a) = \lambda u'(a) \right\}, \\ H_{m+\alpha}^\lambda u &= H_{m+\alpha}u, \end{aligned} \quad (69)$$

where  $\lambda = 2a\tilde{\lambda}/(2a - \tilde{\lambda})$ .

Note that upon integrating by parts it follows that if  $\lambda \geq 0$ , then  $\langle H_{m+\alpha}^\lambda u, u \rangle \geq 0$ , for all  $u \in \text{dom } H_{m+\alpha}^\lambda$ , that is, the self-adjoint operator  $H_{m+\alpha}^\lambda$  is non-negative and, therefore,  $\sigma(H_{m+\alpha}^\lambda) \subset [0, \infty)$ ; from now on we assume that  $\lambda \geq 0$ .

Since each sector is invariant under  $H$ , the Robin self-adjoint extension of full operator  $H$  is given by

$$H^\lambda = \bigoplus_{m \in \mathbb{Z}} H_{m+\alpha}^\lambda \otimes \mathbf{1}, \quad (70)$$

and note that this extension is a special case of Example 4, and the choices of constant functions  $f, g$  guarantee that such self-adjoint extensions preserve angular momentum. The principal cases are for  $\lambda = 0$  and  $\lambda = \infty$ , which correspond to the well-known self-adjoint extensions of Dirichlet and Neumann, respectively.

### 3.3 Scattering for the Robin extensions

In this subsection we study the scattering for Robin self-adjoint realizations of the initial AB operator  $H$  introduced in equation (1). We find the scattering operator  $S$ , we prove the existence of wave operators and that they are complete, and we also obtain explicit expressions for them. In addition, we determined the scattering amplitude and hence the differential scattering cross section for such extensions.

We underline that there is no magnetic field in case  $\alpha = 0$ , and the scattering is sole due to the presence of the solenoid; in fact, the Aharonov-Bohm effect is noticed by comparing the results for  $\alpha \neq 0$  with this reference case  $\alpha = 0$ .

#### 3.3.1 Scattering operator

Assume initially that the wave operators exist and are complete; under such conditions, we shall write out the scattering operator. The solution to

$$\left( -\frac{d^2}{dr^2} - \frac{1}{r} \frac{d}{dr} + \frac{(m+\alpha)^2}{r^2} \right) \varphi = k^2 \varphi, \quad (71)$$

with the linear combination  $\varphi - \lambda \frac{d\varphi}{dr}$  vanishing at  $r = a$ , where  $\lambda = \frac{2a\tilde{\lambda}}{2a - \tilde{\lambda}}$ , is given by

$$\begin{aligned} \varphi_m^\lambda(k, r) = G_m^\lambda(k, a) & \left[ \left( N_{|m+\alpha|}(ka) - \lambda N'_{|m+\alpha|}(ka) \right) J_{|m+\alpha|}(kr) \right. \\ & \left. - \left( J_{|m+\alpha|}(ka) - \lambda J'_{|m+\alpha|}(ka) \right) N_{|m+\alpha|}(kr) \right], \quad (72) \end{aligned}$$



with  $G_m^\lambda(k, a)$  to be determined by imposing condition (43). By using the asymptotic behaviour of  $J_\nu$  and  $N_\nu$  for  $r \rightarrow \infty$ , we obtain

$$\begin{aligned} \varphi_m^\lambda(k, r) &\sim \left(\frac{2}{\pi kr}\right)^{1/2} G_m^\lambda(k, a) \\ &\times \left[ \cos\left(kr - \frac{1}{2}|m + \alpha|\pi - \frac{\pi}{4}\right) \left(N_{|m+\alpha|}(ka) - \lambda N'_{|m+\alpha|}(ka)\right) \right. \\ &\left. - \sin\left(kr - \frac{1}{2}|m + \alpha|\pi - \frac{\pi}{4}\right) \left(J_{|m+\alpha|}(ka) - \lambda J'_{|m+\alpha|}(ka)\right) \right]. \end{aligned} \quad (73)$$

By comparing the above expression with (43) we have

$$\begin{aligned} G_m^\lambda(k, a) &\left[ \cos\left(kr - \frac{1}{2}|m + \alpha|\pi - \frac{\pi}{4}\right) \left(N_{|m+\alpha|}(ka) - \lambda N'_{|m+\alpha|}(ka)\right) \right. \\ &\left. - \sin\left(kr - \frac{1}{2}|m + \alpha|\pi - \frac{\pi}{4}\right) \left(J_{|m+\alpha|}(ka) - \lambda J'_{|m+\alpha|}(ka)\right) \right] \\ &= \cos\left(kr - \frac{1}{2}|m|\pi - \frac{\pi}{4} + \delta_m^\lambda(k, \alpha)\right), \end{aligned} \quad (74)$$

that is,

$$\cos\left(kr - \frac{1}{2}|m + \alpha|\pi - \frac{\pi}{4} + \theta_\lambda\right) = \cos\left(kr - \frac{1}{2}|m|\pi - \frac{\pi}{4} + \delta_m^\lambda(k, \alpha)\right), \quad (75)$$

with  $\theta_\lambda$  so that

$$\cos \theta_\lambda = \frac{N_{|m+\alpha|}(ka) - \lambda N'_{|m+\alpha|}(ka)}{D}, \quad \sin \theta_\lambda = \frac{J_{|m+\alpha|}(ka) - \lambda J'_{|m+\alpha|}(ka)}{D}, \quad (76)$$

with

$$D = \sqrt{\left(N_{|m+\alpha|}(ka) - \lambda N'_{|m+\alpha|}(ka)\right)^2 + \left(J_{|m+\alpha|}(ka) - \lambda J'_{|m+\alpha|}(ka)\right)^2} \quad (77)$$

and, therefore, (43) is satisfied if

$$G_m^\lambda(k, a) = \frac{1}{D}. \quad (78)$$

Note that  $D$  never vanishes. In fact, suppose that  $D = 0$ . Then  $N_{|m+\alpha|}(ka) - \lambda N'_{|m+\alpha|}(ka) = 0$  and  $J_{|m+\alpha|}(ka) - \lambda J'_{|m+\alpha|}(ka) = 0$ . So, it follows that  $J_{|m+\alpha|}(ka)N'_{|m+\alpha|}(ka) - N_{|m+\alpha|}(ka)J'_{|m+\alpha|}(ka) = 0$ . But this is a contradiction with the Wronskian  $W_{r=a}[J_{|m+\alpha|}(kr), N_{|m+\alpha|}(kr)] = 2/(\pi a) \neq 0$ .

Now, comparing the arguments of the cosines above, it is found that the phase shift  $\delta_m^\lambda(k, \alpha)$  is given by

$$\delta_m^\lambda(k, \alpha) = \Delta_m(\alpha) + \theta_\lambda, \quad (79)$$

where  $\Delta_m(\alpha) = \frac{\pi}{2} (|m| - |m + \alpha|)$ . Therefore, the scattering operator  $S_{\alpha, m}^\lambda : \mathcal{H}_k \rightarrow \mathcal{H}_k$  for the Robin self-adjoint extension is

$$S_{\alpha, m}^\lambda = e^{2i\delta_m^\lambda(k, \alpha)} = -e^{2i\Delta_m(\alpha)} \times \left[ \frac{(J_{|m+\alpha|}(ka) - iN_{|m+\alpha|}(ka)) - \lambda (J'_{|m+\alpha|}(ka) - iN'_{|m+\alpha|}(ka))}{(J_{|m+\alpha|}(ka) + iN_{|m+\alpha|}(ka)) - \lambda (J'_{|m+\alpha|}(ka) + iN'_{|m+\alpha|}(ka))} \right], \quad (80)$$

that is,

$$S_{\alpha, m}^\lambda = -e^{2i\Delta_m(\alpha)} \left[ \frac{H_{|m+\alpha|}^{(2)}(ka) - \lambda H_{|m+\alpha|}^{(2)'}(ka)}{H_{|m+\alpha|}^{(1)}(ka) - \lambda H_{|m+\alpha|}^{(1)'}(ka)} \right], \quad (81)$$

where  $H_\nu^{(1), (2)}(x) = J_\nu(x) \pm iN_\nu(x)$  are the Hankel functions [1, 10].

**Remark 3.** Note that for  $\lambda = 0$  we get the Dirichlet case and recover the expression of the scattering operator found in [20]; and, if we choose  $\lambda = \infty$  we obtain the scattering operator for the Neumann case, namely,

$$S_{\alpha, m}^\mathcal{N} = -e^{2i\Delta_m(\alpha)} \frac{H_{|m+\alpha|}^{(2)'}(ka)}{H_{|m+\alpha|}^{(1)'}(ka)}. \quad (82)$$

### 3.3.2 Asymptotic behaviours of the scattering operator

Now we describe the asymptotic behaviour of the scattering operator for different self-adjoint extensions for both  $ka \rightarrow \infty$  and  $ka \rightarrow 0$ , and also compare the results. This is done from the asymptotic behaviour of Bessel functions.

We begin with the behaviour for  $ka \rightarrow \infty$ . For this we recall that

$$J'_{|m+\alpha|}(ka) = \left( -J_{|m+\alpha|+1}(ka)k + \frac{|m+\alpha|}{a} J_{|m+\alpha|}(ka) \right), \quad (83)$$

and thus, its behaviour for  $ka \rightarrow \infty$  is given by

$$J'_{|m+\alpha|}(ka) \sim - \left( \frac{2}{\pi ka} \right)^{1/2} \cos \left( ka - (|m+\alpha| + 1) \frac{\pi}{2} - \frac{\pi}{4} \right) k + \frac{|m+\alpha|}{a} \left( \frac{2}{\pi ka} \right)^{1/2} \cos \left( ka - |m+\alpha| \frac{\pi}{2} - \frac{\pi}{4} \right); \quad (84)$$

and so

$$J'_{|m+\alpha|}(ka) \sim - \left( \frac{2}{\pi ka} \right)^{1/2} \sin \left( ka - |m+\alpha| \frac{\pi}{2} - \frac{\pi}{4} \right) k + \frac{|m+\alpha|}{a} \left( \frac{2}{\pi ka} \right)^{1/2} \cos \left( ka - |m+\alpha| \frac{\pi}{2} - \frac{\pi}{4} \right). \quad (85)$$

Similarly,

$$N'_{|m+\alpha|}(ka) = \left( -N_{|m+\alpha|+1}(ka)k + \frac{|m+\alpha|}{a} N_{|m+\alpha|}(ka) \right), \quad (86)$$

and its behaviour for  $ka \rightarrow \infty$  is given by

$$N'_{|m+\alpha|}(ka) \sim \left( \frac{2}{\pi ka} \right)^{1/2} \cos \left( ka - |m+\alpha| \frac{\pi}{2} - \frac{\pi}{4} \right) k + \frac{|m+\alpha|}{a} \left( \frac{2}{\pi ka} \right)^{1/2} \sin \left( ka - |m+\alpha| \frac{\pi}{2} - \frac{\pi}{4} \right). \quad (87)$$

Now, by considering  $\lambda \neq 0$ , the asymptotic behaviour of Bessel functions and their derivatives given above, we get for  $ka \rightarrow \infty$

$$S_{\alpha,m}^{\lambda} \sim (-1)^m e^{-2ika+i\pi/2} \times \left[ \frac{(ka)^2 \lambda^2 + 2i\lambda(\lambda|m+\alpha| - a)ka - \lambda^2|m+\alpha|^2 + 2\lambda|m+\alpha|a - a^2}{(ka)^2 \lambda^2 + \lambda^2|m+\alpha|^2 - 2\lambda|m+\alpha| + a^2} \right], \quad (88)$$

and, since the term in square brackets is approximately 1 in this case, we have

$$S_{\alpha,m}^{\lambda} \sim (-1)^m e^{-2ika+i\pi/2}, \quad (89)$$

for  $ka \rightarrow \infty$ . This expression coincides with the Neumann case, i.e., the scattering operator for the Robin case acts as the Neumann case for large energy, independently of  $\lambda$ , provided that  $\lambda \neq 0$ . However, it differs from the Dirichlet case

$$S_{\alpha,m}^{\mathcal{D}} \sim (-1)^m e^{-2ika-i\pi/2}, \quad (90)$$

for  $ka \rightarrow \infty$ .

Summing up, for very large energies the scattering operator does not distinguish different Robin extensions (i.e.,  $0 < \lambda \leq \infty$ ) from the Neumann case  $\lambda = \infty$ , but it has a different behaviour from the Dirichlet case  $\lambda = 0$ . In order to try to understand such behaviour intuitively, let us informally consider the perhaps simplest situation, that is, the “free” unidimensional reflection from a barrier at the origin  $x = 0$  with wavefunction  $\psi(x) = A(k) \sin(kx) + B(k) \cos(kx)$ , at least near the origin; Dirichlet and Neumann boundary conditions impose that  $B = 0$  and  $A = 0$ ,

respectively, whereas Robin condition  $\psi(0) = \lambda\psi'(0)$  imposes the energy relation  $|\lambda k|^2 = |B(k)|^2/|A(k)|^2$ , and for large energies  $k \rightarrow \infty$  one has  $|A(k)| \ll |B(k)|$  and in this region the system behaviour becomes similar to the one dictated by the Neumann condition.

On the other hand, taking into account that the behaviour of the Bessel functions for  $ka \rightarrow 0$  [16] are given by

$$J_{|m+\alpha|}(ka) \sim [(1/2)ka]^{|m+\alpha|}/\Gamma(|m+\alpha|+1), \quad (91)$$

and

$$N_{|m+\alpha|}(ka) \sim -\frac{\Gamma(|m+\alpha|)}{\pi[(1/2)ka]^{|m+\alpha|}}, \quad (92)$$

we get the following behaviour for the scattering operator  $S_{\alpha,m}^\lambda$  for  $ka \rightarrow 0$

$$\begin{aligned} S_{\alpha,m}^\lambda \sim & \cos \beta \frac{d_1^2 - d_2(ka/2)^{4|m+\alpha|}}{d_1^2 + d_2(ka/2)^{4|m+\alpha|}} - \sin \beta \frac{2d_1d_3(ka/2)^{2|m+\alpha|}}{d_1^2 + d_2(ka/2)^{4|m+\alpha|}} \\ & + i \left[ \sin \beta \frac{d_1^2 - d_2(ka/2)^{4|m+\alpha|}}{d_1^2 + d_2(ka/2)^{4|m+\alpha|}} + \cos \beta \frac{2d_1d_3(ka/2)^{2|m+\alpha|}}{d_1^2 + d_2(ka/2)^{4|m+\alpha|}} \right], \end{aligned} \quad (93)$$

with  $\beta = \pi(|m| - |m+\alpha|)$  and the coefficients

$$\begin{aligned} d_1 &:= -\Gamma(|m+\alpha|)/\pi - \lambda[2\Gamma(|m+\alpha|+1) - |m+\alpha|\Gamma(|m+\alpha|)]/(\pi a), \\ d_2 &:= \Gamma(|m+\alpha|+1)^{-2}(1 + \lambda^2|m+\alpha|^2/a^2 - 2\lambda|m+\alpha|/a), \\ d_3 &:= \Gamma(|m+\alpha|+1)^{-1}(1 - \lambda|m+\alpha|/a), \end{aligned} \quad (94)$$

are independent of  $k$ . For  $m = 0$  and  $\alpha = 0$  we have

$$\begin{aligned} S_{0,0}^\lambda \sim & \frac{1 - \frac{\pi^2}{4\ln(ka)^2} + \lambda \left( \frac{2}{a\ln(ka)} - \frac{\pi^2(ka/2)^2}{a\ln(ka)^2} \right) + \lambda^2 \left( \frac{1-\pi^2(ka/2)^4}{a^2\ln(ka)^2} \right)}{1 + \frac{\pi^2}{4\ln(ka)^2} + \lambda \left( \frac{2}{a\ln(ka)} + \frac{\pi^2(ka/2)^2}{a\ln(ka)^2} \right) + \lambda^2 \left( \frac{1+\pi^2(ka/2)^4}{a^2\ln(ka)^2} \right)} \\ & + i \frac{\pi}{\ln(ka)} \frac{1 + \lambda \left( \frac{1}{a\ln(ka)} + \frac{2(ka/2)^2}{a} \right) + \lambda^2 \frac{2(ka/2)^2}{a^2\ln(ka)}}{1 + \frac{\pi^2}{4\ln(ka)^2} + \lambda \left( \frac{2}{a\ln(ka)} + \frac{\pi^2(ka/2)^2}{a\ln(ka)^2} \right) + \lambda^2 \left( \frac{1+\pi^2(ka/2)^4}{a^2\ln(ka)^2} \right)}, \end{aligned} \quad (95)$$

for  $ka \rightarrow 0$ .

We observed that, for very small energies  $ka \rightarrow 0$ , the sole scattering operator is not able to distinguish the Robin, Dirichlet and Neumann self-adjoint extensions of the initial AB hamiltonian (1). Furthermore, this occurs both for the case with field ( $\alpha \neq 0$ ) and for the reference case without field ( $\alpha = 0$ ). These behaviours were numerically recovered.

### 3.3.3 Wave operators

Now, we prove that in fact the wave operators for the Robin self-adjoint extensions exist and are complete. In addition, we obtain an explicit expression for them. We begin with a lemma that will be used in the proof of Theorem 3.

**Lemma 1.** *Let  $A$  and  $B$  be self-adjoint operators and  $U$  a bounded operator so that  $AU = UB$ , assuming that the compositions are well defined. Then*

$$e^{-iAt}U = Ue^{-iBt}. \quad (96)$$

*Proof.* For each  $\xi \in \text{dom } B$  so that  $U\xi \in \text{dom } A$ , let  $u(t) = Ue^{-iBt}\xi$  and  $v(t) = e^{-iAt}U\xi$ , for all  $t \in \mathbb{R}$ . We want to show that  $u(t) = v(t)$ , for all  $t \in \mathbb{R}$ . If  $w(t) = \|v(t) - u(t)\|^2$ , then

$$\begin{aligned} \frac{dw}{dt} &= \frac{d}{dt} \langle v(t) - u(t), v(t) - u(t) \rangle = 2\text{Re} \left\langle v(t) - u(t), \frac{d}{dt}[v(t) - u(t)] \right\rangle \\ &= 2\text{Re} \left[ (-i) \left( \langle e^{-iAt}U\xi, Ae^{-iAt}U\xi \rangle + \langle Ue^{-iBt}\xi, AUe^{-iBt}\xi \rangle \right. \right. \\ &\quad \left. \left. - 2\text{Re} \langle Ae^{-iAt}U\xi, Ue^{-iBt}\xi \rangle \right) \right] = 0 \end{aligned} \quad (97)$$

because what is in square brackets is real since  $A$  is self-adjoint. Therefore  $w(t)$  is constant. Since  $w(0) = 0$ , it follows that  $w(t) = 0$ , for all  $t \in \mathbb{R}$ .  $\square$

Let  $P_a : \mathcal{H}_r \rightarrow \mathcal{H}_r^a$  be given by  $(P_a\psi)(r) = (\chi_{[a,\infty)}\psi)(r)$ , that is,  $P_a$  is the orthogonal projection operator onto  $\mathcal{H}_r^a$ .

**Theorem 3.** *The wave operators*

$$\mathcal{W}_{\pm, \alpha, m}^\lambda = \text{s-} \lim_{t \rightarrow \pm\infty} e^{iH_{m+\alpha}^\lambda t} P_a \mathcal{F}_m e^{-ik^2 t} \quad (98)$$

*exist and are surjective isometries from  $\mathcal{H}_k$  onto  $\mathcal{H}_r^a$ . Explicitly, for  $\psi \in \mathcal{H}_k$ , they are given by the expressions*

$$\left( \mathcal{W}_{\pm, \alpha, m}^\lambda \psi \right) (r) = i^{|m|} \lim_{R \rightarrow \infty} \int_0^R \varphi_m^\lambda(k, r) e^{\mp i \delta_m^\lambda(k, \alpha)} \psi(k) k dk. \quad (99)$$

*Proof.* We consider only the case  $\mathcal{W}_{-, \alpha, m}^\lambda$ ; the proof for  $\mathcal{W}_{+, \alpha, m}^\lambda$  is similar. Since the proof is rather long, we divide it in three steps:

*1<sup>st</sup> Step:* Define the candidate for the limit operator, and show two equalities.

*2<sup>nd</sup> Step:* Show that the wave operator  $\mathcal{W}_{-, \alpha, m}^\lambda$  exists and satisfies (99).

*3<sup>rd</sup> Step:* Show that the wave operator  $\mathcal{W}_{-, \alpha, m}^\lambda$  is a surjective isometry.

1<sup>st</sup> Step: “Define the candidate for the limit operator, and show two equalities.” Let us define an operator  $U_{-, \alpha, m}^\lambda$  by

$$\left( U_{-, \alpha, m}^\lambda \psi \right) (r) = i^{|m|} \int_\varepsilon^R \varphi_m^\lambda(k, r) e^{i\delta_m^\lambda(k, \alpha)} \psi(k) k dk, \quad (100)$$

with  $\psi \in \mathcal{H}_k$  and  $\text{supp } \psi \subset (\varepsilon, R)$ . This operator is well defined by Hölder’s inequality. By using that

$$J_{|\nu|}(kr) = (2/\pi)^{1/2} \cos \left( kr - |\nu|\pi/2 - \pi/4 \right) / (kr)^{1/2} + O((kr)^{-3/2}) \quad (101)$$

and

$$N_{|\nu|}(kr) = (2/\pi)^{1/2} \sin \left( kr - |\nu|\pi/2 - \pi/4 \right) / (kr)^{1/2} + O((kr)^{-3/2}), \quad (102)$$

it is found that  $U_{-, \alpha, m}^\lambda \psi \in \mathcal{H}_r^a$ , and it satisfies

$$\left\| U_{-, \alpha, m}^\lambda \psi \right\|_{\mathcal{H}_r^a} \leq K_\lambda(R, \varepsilon) \|\psi\|_{\mathcal{H}_k}, \quad (103)$$

where  $K_\lambda(R, \varepsilon)$  depends only on  $R$  and  $\varepsilon$ .

Now we check that  $U_{-, \alpha, m}^\lambda \psi$  is actually in the domain of  $H_{m+\alpha}^\lambda$  and

$$H_{m+\alpha}^\lambda U_{-, \alpha, m}^\lambda \psi = U_{-, \alpha, m}^\lambda k^2 \psi. \quad (104)$$

In fact, let  $u \in \text{dom } H_{m+\alpha}^\lambda$ . Then, by Fubini’s theorem, we can write

$$\left\langle H_{m+\alpha}^\lambda u, U_{-, \alpha, m}^\lambda \psi \right\rangle = \int_a^\infty \overline{(H_{m+\alpha}^\lambda u)(r)} (U_{-, \alpha, m}^\lambda \psi)(r) r dr \quad (105)$$

$$\begin{aligned} &= \int_a^\infty \left[ \left( -\frac{d^2}{dr^2} - \frac{1}{r} \frac{d}{dr} + \frac{(m+\alpha)^2}{r^2} \right) \overline{u}(r) i^{|m|} \right. \\ &\quad \left. \times \int_\varepsilon^R \varphi_m^\lambda(k, r) e^{i\delta_m^\lambda(k, \alpha)} \psi(k) k dk \right] r dr \quad (106) \end{aligned}$$

$$\begin{aligned} &= i^{|m|} \int_\varepsilon^R e^{i\delta_m^\lambda(k, \alpha)} \psi(k) \left[ (m+\alpha)^2 \int_a^\infty \frac{1}{r^2} \varphi_m^\lambda(k, r) \overline{u}(r) r dr \right. \\ &\quad \left. - \int_a^\infty \varphi_m^\lambda(k, r) \left( \frac{d^2}{dr^2} + \frac{1}{r} \frac{d}{dr} \right) \overline{u}(r) r dr \right] k dk, \quad (107) \end{aligned}$$

and integrating by parts the second term in square brackets, we obtain

$$\begin{aligned} \left\langle H_{m+\alpha}^\lambda u, U_{-, \alpha, m}^\lambda \psi \right\rangle &= i^{|m|} \int_\varepsilon^R e^{i\delta_m^\lambda(k, \alpha)} \psi(k) \\ &\quad \times \left[ \int_a^\infty \left( -\frac{d^2}{dr^2} - \frac{1}{r} \frac{d}{dr} + \frac{(m+\alpha)^2}{r^2} \right) \varphi_m^\lambda(k, r) \overline{u}(r) r dr \right] k dk; \quad (108) \end{aligned}$$

note that there are no boundary terms since  $\varphi_m^\lambda(k, a) = \lambda \frac{d\varphi_m^\lambda}{dr}(k, a)$  and  $u(a) = \lambda u'(a)$ . Therefore,

$$\left\langle H_{m+\alpha}^\lambda u, U_{-, \alpha, m}^\lambda \psi \right\rangle = i^{|m|} \int_\varepsilon^R e^{i\delta_m^\lambda(k, \alpha)} \psi(k) \left[ \int_a^\infty k^2 \varphi_m^\lambda(k, r) \bar{u}(r) r dr \right] k dk, \quad (109)$$

and, again by Fubini, we find that

$$\left\langle H_{m+\alpha}^\lambda u, U_{-, \alpha, m}^\lambda \psi \right\rangle = \int_a^\infty \bar{u}(r) i^{|m|} \int_\varepsilon^R \varphi_m^\lambda(k, r) e^{i\delta_m^\lambda(k, \alpha)} k^2 \psi(k) k dk r dr, \quad (110)$$

that is,

$$\left\langle H_{m+\alpha}^\lambda u, U_{-, \alpha, m}^\lambda \psi \right\rangle = \left\langle u, U_{-, \alpha, m}^\lambda k^2 \psi \right\rangle, \quad (111)$$

where  $\langle \cdot, \cdot \rangle$  denotes the inner product in  $\mathcal{H}_r^a$ . Then  $U_{-, \alpha, m}^\lambda \psi \in \text{dom } H_{m+\alpha}^\lambda$  and

$$H_{m+\alpha}^\lambda U_{-, \alpha, m}^\lambda \psi = U_{-, \alpha, m}^\lambda k^2 \psi. \quad (112)$$

Finally, apply Lemma 1 to conclude

$$e^{-iH_{m+\alpha}^\lambda t} U_{-, \alpha, m}^\lambda \psi = U_{-, \alpha, m}^\lambda e^{-ik^2 t} \psi. \quad (113)$$

*2<sup>nd</sup> Step:* “Show that the wave operator exists and satisfies (99).” Assume that  $\psi \in C_0^\infty(\varepsilon, R)$ . By using the conclusion of the first step (i.e., the last equality above), one can write,

$$\left\| e^{iH_{m+\alpha}^\lambda t} P_a \mathcal{F}_m e^{-ik^2 t} \psi - U_{-, \alpha, m}^\lambda \psi \right\|^2 = \left\| P_a \mathcal{F}_m e^{-ik^2 t} \psi - e^{-iH_{m+\alpha}^\lambda t} U_{-, \alpha, m}^\lambda \psi \right\|^2 \quad (114)$$

$$= \left\| P_a \mathcal{F}_m e^{-ik^2 t} \psi - U_{-, \alpha, m}^\lambda e^{-ik^2 t} \psi \right\|^2 \quad (115)$$

$$= \int_a^\infty \left| \left( (P_a \mathcal{F}_m - U_{-, \alpha, m}^\lambda) e^{-ik^2 t} \psi \right) (r) \right|^2 r dr \quad (116)$$

$$= \int_a^\infty \left| i^{|m|} \int_0^\infty J_{|m|}(kr) e^{-ik^2 t} \psi(k) k dk - i^{|m|} \int_\varepsilon^R dk k \varphi_m^\lambda(k, r) e^{i\delta_m^\lambda(k, \alpha)} e^{-ik^2 t} \psi(k) k dk \right|^2 r dr \quad (117)$$

$$= \int_a^\infty \left| i^{|m|} \int_\varepsilon^R e^{-ik^2 t} \psi(k) \left( J_{|m|}(kr) - \varphi_m^\lambda(k, r) e^{i\delta_m^\lambda(k, \alpha)} \right) k dk \right|^2 r dr, \quad (118)$$

and after some calculations with the asymptotic behaviour of the two functions in brackets above, we obtain

$$\begin{aligned} & \left\| e^{iH_{m+\alpha}^\lambda t} P_a \mathcal{F}_m e^{-ik^2 t} \psi - U_{-, \alpha, m}^\lambda \psi \right\|^2 \\ &= \int_a^\infty \left| \int_\varepsilon^R e^{-ik^2 t} \psi(k) \left[ K_1^\lambda(ka) \frac{e^{ikr}}{(kr)^{1/2}} + K_2^\lambda(ka) O\left((kr)^{-3/2}\right) \right] k dk \right|^2 r dr, \end{aligned} \quad (119)$$

with  $K_1^\lambda$  and  $K_2^\lambda$  are functions of class  $C^\infty$ . Finally, using the inequality  $\|f + g\|^2 \leq 2\|f\|^2 + 2\|g\|^2$ , we get

$$\begin{aligned} & \left\| e^{iH_{m+\alpha}^\lambda t} P_a \mathcal{F}_m e^{-ik^2 t} \psi - U_{-, \alpha, m}^\lambda \psi \right\|^2 \\ & \leq 2 \int_a^\infty \left| \int_\varepsilon^R e^{-ik^2 t} \psi(k) K_1^\lambda(ka) \frac{e^{ikr}}{(kr)^{1/2}} k dk \right|^2 r dr \\ & \quad + 2 \int_a^\infty \left| \int_\varepsilon^R e^{-ik^2 t} \psi(k) K_2^\lambda(ka) O\left((kr)^{-3/2}\right) k dk \right|^2 r dr. \end{aligned} \quad (120)$$

We discuss each term on the right side of the last inequality separately. By replacing  $e^{-ik^2 t + ikr}$  with  $(-2ikt + ir)^{-1} \partial_k e^{-ik^2 t + ikr}$ , integrating by parts and using the dominated convergence theorem to estimate the first term, it is found that it vanishes as  $t \rightarrow -\infty$ .

For the second term, let  $h(kr) = O\left((kr)^{-3/2}\right)$ , then  $h(kr) = M(kr) \times (kr)^{-3/2}$ , with  $M(kr)$  a bounded function. So, by Riemann-Lebesgue lemma and dominated convergence theorem,

$$\begin{aligned} & \int_a^\infty \left| \int_\varepsilon^R e^{-ik^2 t} \psi(k) K_2^\lambda(ka) O\left((kr)^{-3/2}\right) k dk \right|^2 r dr \\ &= \int_a^\infty \left| \int_\varepsilon^R e^{-ik^2 t} \psi(k) K_2^\lambda(ka) M(kr) k^{-1/2} dk \right|^2 r^{-2} dr \rightarrow 0, \end{aligned} \quad (121)$$

as  $t \rightarrow -\infty$ . Then

$$\left\| e^{iH_{m+\alpha}^\lambda t} P_a \mathcal{F}_m e^{-ik^2 t} \psi - U_{-, \alpha, m}^\lambda \psi \right\| \rightarrow 0, \quad (122)$$

as  $t \rightarrow -\infty$ , and since  $C_0^\infty(0, \infty)$  is dense in  $\mathcal{H}_k$ , it follows that the wave operator  $\mathcal{W}_{-, \alpha, m}^\lambda$  exists and satisfies (99).

*3<sup>rd</sup> Step:* “Show that the wave operator  $\mathcal{W}_{-, \alpha, m}^\lambda$  is a surjective isometry.” To show that the wave operator is an isometry we take  $\psi \in \mathcal{H}_k$  with compact support and check

$$\left\| \mathcal{W}_{-, \alpha, m}^\lambda \psi \right\| = \lim_{t \rightarrow -\infty} \left\| P_a \mathcal{F}_m e^{-ik^2 t} \psi \right\| = \lim_{t \rightarrow -\infty} \left\| \mathcal{F}_m e^{-ik^2 t} \psi \right\| = \|\psi\|. \quad (123)$$



To prove that  $\text{rng } \mathcal{W}_{-, \alpha, m}^\lambda = \mathcal{H}_r^a$  it suffices to show that its adjoint is an isometry, because the kernel  $\{0\} = \text{N}((\mathcal{W}_{-, \alpha, m}^\lambda)^*) = (\text{rng } \mathcal{W}_{-, \alpha, m}^\lambda)^\perp$ , and so  $\text{rng } \mathcal{W}_{-, \alpha, m}^\lambda = \mathcal{H}_r^a$ . Since  $\mathcal{W}_{-, \alpha, m}^\lambda (\mathcal{W}_{-, \alpha, m}^\lambda)^*$  is the orthogonal projection onto  $\text{rng } \mathcal{W}_{-, \alpha, m}^\lambda$ , which is closed, and since

$$\begin{aligned}
& \left[ \mathcal{W}_{-, \alpha, m}^\lambda \left( (\mathcal{W}_{-, \alpha, m}^\lambda)^* \psi \right) (k) \right] (r) \\
&= i^{|m|} \lim_{R \rightarrow \infty} \int_0^R \varphi_m^\lambda(k, r) e^{i\delta_m^\lambda(k, \alpha)} \left( (\mathcal{W}_{-, \alpha, m}^\lambda)^* \psi \right) (k) k dk \\
&= i^{|m|} \lim_{R \rightarrow \infty} \int_0^R \varphi_m^\lambda(k, r) e^{i\delta_m^\lambda(k, \alpha)} \left( (-i)^{|m|} \int_a^\infty \varphi_m^\lambda(k, s) e^{-i\delta_m^\lambda(k, \alpha)} \psi(s) s ds \right) k dk \\
&= \lim_{R \rightarrow \infty} \int_0^R G_m^\lambda(k, a) D_\nu^\lambda(ka, kr) \int_a^\infty G_m^\lambda(k, a) D_\nu^\lambda(ka, ks) \psi(s) s ds k dk,
\end{aligned} \tag{124}$$

and recalling that  $G_m^\lambda(k, a) = \frac{1}{D}$ , with

$$D = \sqrt{(N_\nu(ka) - \lambda N'_\nu(ka))^2 + (J_\nu(ka) - \lambda J'_\nu(ka))^2}, \tag{125}$$

and  $\nu = |m + \alpha|$ , one has

$$D_\nu^\lambda(ka, y) := [N_\nu(ka) - \lambda N'_\nu(ka)] J_\nu(y) - [J_\nu(ka) - \lambda J'_\nu(ka)] N_\nu(y), \tag{126}$$

and, in order to conclude the theorem, it is enough to prove the following lemma.

**Lemma 2.** *Let  $\psi \in C_0^\infty(a, \infty)$  and  $\nu \geq 0$ . Then*

$$\psi(r) = \lim_{R \rightarrow \infty} \int_{1/R}^R \frac{1}{D^2} D_\nu^\lambda(ka, kr) \int_a^\infty D_\nu^\lambda(ka, ks) \psi(s) s ds k dk. \tag{127}$$

In fact, this lemma implies that

$$\left[ \mathcal{W}_{-, \alpha, m}^\lambda \left( (\mathcal{W}_{-, \alpha, m}^\lambda)^* \psi \right) (k) \right] (r) = \psi(r), \tag{128}$$

for all  $\psi \in C_0^\infty(a, \infty)$ , and since this set is dense in  $\mathcal{H}_r^a$ , it follows that  $\text{rng } \mathcal{W}_{-, \alpha, m}^\lambda = \mathcal{H}_r^a$ , and the theorem is proved.  $\square$

In the following we present the proof of Lemma 2.

*Proof.* Since the Wronskian of  $J_\nu(z)$  and  $N_\nu(z)$  is equal to  $2/(\pi z)$  [16], that is,  $W_z[J_\nu, N_\nu] = 2/(\pi z)$ , one has  $W_r[J_\nu(kr), N_\nu(kr)] = 2/(\pi r)$ . Now, we consider the boundary value problem

$$\begin{aligned}
(E - H_\nu)\varphi &= \psi, \quad a < r < \infty, \quad |\text{Im } E| > 0, \\
\varphi(a) - \lambda \varphi'(a) &= 0.
\end{aligned} \tag{129}$$

The Green's function

$$g(r|s) = \begin{cases} \frac{u_1(r)u_2(s)}{W_s[u_1, u_2]}, & a < r < s, \\ \frac{u_1(s)u_2(r)}{W_s[u_1, u_2]}, & s < r < \infty, \end{cases} \quad (130)$$

is the solution to the auxiliary problem

$$\begin{aligned} (E - H_\nu)g &= \delta(r - s), \quad a < r < \infty, \\ g(a) - \lambda g'(a) &= 0, \end{aligned} \quad (131)$$

where

$$\begin{aligned} u_1(r) &= \left[ N_\nu(E^{1/2}a) - \lambda N'_\nu(E^{1/2}a) \right] J_\nu(E^{1/2}r) \\ &\quad - \left[ J_\nu(E^{1/2}a) - \lambda J'_\nu(E^{1/2}a) \right] N_\nu(E^{1/2}r) \\ &= D_\nu^\lambda(E^{1/2}a, E^{1/2}r), \end{aligned} \quad (132)$$

is the solution to  $(E - H_\nu)u = 0$  that satisfies the boundary condition at  $r = a$ , and

$$u_2(r) = H_\nu^{(1),(2)}(E^{1/2}r), \quad (133)$$

is the solution to  $(E - H_\nu)u = 0$  that satisfies the boundary condition at  $\infty$ , and the superscripts (1) and (2) correspond to  $\text{Im } E > 0$  (with  $\text{Im } \sqrt{E} > 0$ ) and  $\text{Im } E < 0$  (with  $\text{Im } \sqrt{E} < 0$ ), respectively;  $W_s[u_1, u_2]$  is the wronskian of the solutions  $u_1$  and  $u_2$  at the point  $r = s$ , and in this case one has

$$W_s[u_1, u_2] = \frac{2}{\pi s} \left( u_2(a) - \lambda \frac{du_2}{dr}(a) \right). \quad (134)$$

Write  $R_E := (E - H_\nu)^{-1}$  for the resolvent of  $H_\nu$  at “energy”  $E$ , so that the solution  $(R_E\psi)(r)$  to problem (129) is given by

$$\begin{aligned} (R_E\psi)(r) &= \int_a^\infty g(r|s)\psi(s)ds \\ &= \int_a^r \frac{u_1(s)u_2(r)}{W_s[u_1, u_2]}\psi(s)ds + \int_r^\infty \frac{u_1(r)u_2(s)}{W_s[u_1, u_2]}\psi(s)ds \\ &= \frac{\pi}{2} \left[ u_2(a) - \lambda \frac{du_2}{dr}(a) \right]^{-1} \left[ H_\nu^{(1),(2)}(E^{1/2}r) \int_a^r D_\nu^\lambda(E^{1/2}a, E^{1/2}s) \right. \\ &\quad \left. \times \psi(s)s ds + D_\nu^\lambda(E^{1/2}a, E^{1/2}r) \int_r^\infty H_\nu^{(1),(2)}(E^{1/2}s)\psi(s)s ds \right]. \end{aligned} \quad (135)$$

Recall now the Stone formula [8] for the spectral projection of  $H_\nu$  onto the interval  $[a, b]$ ,

$$\chi_{[a,b]}(H_\nu) = \text{s-} \lim_{\delta \rightarrow 0^+} \frac{1}{2\pi i} \int_a^b (R_{x-i\delta} - R_{x+i\delta})dx. \quad (136)$$

If one writes  $E_- = x - i\delta$  and  $E_+ = x + i\delta$  for the “energy”  $E$  with  $\text{Im } E < 0$  and  $\text{Im } E > 0$ , respectively, then

$$[\chi_{[1/R, R]}(H_\nu)\psi](r) = \lim_{\delta \rightarrow 0^+} \frac{1}{2\pi i} \int_{1/R}^R [(R_{E_-}\psi)(r) - (R_{E_+}\psi)(r)] dx. \quad (137)$$

Now, substitute the above expressions for the resolvent operators and use the dominated convergence theorem (take into account that the functions  $J_\nu$  and  $N_\nu$  are continuous and bounded), after some manipulations and simplifications we obtain the expression

$$\begin{aligned} & [\chi_{[1/R, R]}(H_\nu)\psi](r) \\ &= \frac{1}{2} \int_{1/R}^R \frac{D_\nu^\lambda(x^{1/2}a, x^{1/2}r)}{(N_\nu(x^{1/2}a) - \lambda N'_\nu(x^{1/2}a))^2 + (J_\nu(x^{1/2}a) - \lambda J'_\nu(x^{1/2}a))^2} \\ & \quad \times \int_a^\infty D_\nu^\lambda(x^{1/2}a, x^{1/2}s)\psi(s)s ds dx. \end{aligned} \quad (138)$$

Finally, use the change of variable  $x^{1/2} = k$ , so that  $dx/2 = k dk$ , to get

$$[\chi_{[1/R, R]}(H_\nu)\psi](r) = \int_{1/R}^R \frac{1}{D^2} D_\nu^\lambda(ka, kr) \int_a^\infty D_\nu^\lambda(ka, ks)\psi(s)s ds k dk, \quad (139)$$

which is a.e. equal to the function defined by the integral on the right side of (127). On the other hand,  $\chi_{[a, b]}(H_\nu) \equiv 0$  for  $[a, b] \subset (-\infty, 0)$  since  $H_\nu$  is a positive operator (see Theorem 8.3.13 in [8]), and so  $\sigma(H_\nu) \subset [0, \infty)$  and  $H_\nu$  has no eigenvalues.  $\square$

### 3.3.4 Scattering amplitude and cross section

In this subsection we calculate the scattering amplitude and differential scattering cross section for the Robin self-adjoint extensions, and some comparisons will be made in the next subsection.

However, first we recall what was done in [20] to determine the scattering amplitude  $f_\alpha$  for the case of a solenoid of radius zero, and with Dirichlet condition at the origin; since we will make use of such results. In the case of radius zero, in each sector of angular momentum  $m$ , one has

$$\Delta_m(\alpha) = \frac{\pi}{2}(|m| - |m + \alpha|), \quad (140)$$

for the phase shift, which is a function only of  $\alpha$ , and so the corresponding scattering operator is

$$e^{2i\Delta_m(\alpha)} = \begin{cases} e^{-i\pi\alpha}, & m \geq -\alpha \\ e^{i\pi\alpha}, & m \leq -\alpha \end{cases}. \quad (141)$$

Then, the Fourier coefficients of  $f_\alpha$  in the expression (46) has constant modulus and do not vanish as  $|m| \rightarrow \infty$ ; so the scattering amplitude  $f_\alpha$  is seen as a distribution. To obtain the correct expression of the amplitude  $f_\alpha$ , note that the scattering operator  $S_\alpha$  on  $\mathcal{H}_0$  is an integral operator and, by using the above expressions, in [20] it was found that

$$(S_\alpha \xi)(k, \theta) = \int_0^{2\pi} s_\alpha(\theta - \theta') \xi(k, \theta') d\theta', \quad (142)$$

with

$$s_\alpha(\theta) = \delta(\theta) \cos(\pi\alpha) + i \frac{\sin(\pi\alpha)}{\pi} \text{PV} \left( \frac{1}{e^{i\theta} - 1} \right), \quad (143)$$

where PV denotes the principal value (recall that here  $0 \leq \alpha < 1$ ). These expressions and the relation  $(S - \mathbf{1})(k, \theta) = \left(\frac{ik}{2\pi}\right)^{1/2} f(k, \theta)$  imply the following expression for the scattering amplitude

$$f_\alpha(k, \theta) = \left(\frac{2\pi}{ik}\right)^{1/2} \left[ \delta(\theta) [\cos(\pi\alpha) - 1] + i \frac{\sin(\pi\alpha)}{\pi} \text{PV} \left( \frac{1}{e^{i\theta} - 1} \right) \right]. \quad (144)$$

Now, if  $\theta \neq 0$  the distribution  $f_\alpha$  is represented by the function [20]

$$f_\alpha(k, \theta) = \frac{\sin(\pi\alpha)}{(2\pi ik)^{1/2}} \frac{e^{-i\theta/2}}{\sin(\theta/2)}, \quad (145)$$

and so the differential scattering cross section in this case is

$$\left( \frac{d\sigma}{d\theta} \right)_\alpha(k, \theta) = \frac{1}{2\pi k} \frac{\sin^2(\pi\alpha)}{\sin^2(\theta/2)}, \quad \theta \neq 0, \quad (146)$$

which agree with the expressions found by Aharonov and Bohm [4] and also by other authors, for example in [14]. Thus we will continue looking at the scattering amplitude as a distribution, which will be conveniently calculated from the Fourier series.

**Remark 4.** We observe that in [14] it is advocated that there should be no  $\delta(\theta)$  in the above expression (144) for the scattering amplitude, and that  $f_\alpha$  should be restricted to (145); this causes a controversy with references [14] and [20]. In any event, since we do not consider the forward direction  $\theta = 0$  in our comparisons of the scattering due to different self-adjoint extensions (i.e., our main goal in the next section), we are able to keep away from such controversy.

Now we turn to our Robin extensions and positive radius. For  $\alpha = 0$ , that is, no magnetic field, the expression (46) gives for scattering amplitude associated with  $H^\lambda$

$$f_0^\lambda(k, \theta) = \frac{1}{(2\pi ik)^{1/2}} \sum_{m=-\infty}^{\infty} \left( e^{2i\delta_m^\lambda(k,0)} - 1 \right) e^{im\theta}, \quad (147)$$

and since

$$e^{2i\delta_m^\lambda(k,0)} = -\frac{H_{|m|}^{(2)}(ka) - \lambda H_{|m|}^{(2)'}(ka)}{H_{|m|}^{(1)}(ka) - \lambda H_{|m|}^{(1)'}(ka)}, \quad (148)$$

we obtain

$$f_0^\lambda(k, \theta) = -\left(\frac{2}{\pi i k}\right)^{1/2} \sum_{m=-\infty}^{\infty} \frac{J_{|m|}(ka) - \lambda J_{|m|}'(ka)}{H_{|m|}^{(1)}(ka) - \lambda H_{|m|}^{(1)'}(ka)} e^{im\theta}. \quad (149)$$

Note that for fixed  $ka$  and  $k \neq 0$ , the series above is convergent since its coefficients are fast decaying as  $|m| \rightarrow \infty$ , due to the well-known behaviour of Bessel functions. Thus, in this case  $f_0^\lambda$  is represented by a function and therefore the differential cross section is given by

$$\left(\frac{d\sigma}{d\theta}\right)_0^\lambda(k, \theta) = \frac{2}{\pi k} \left| \sum_{m=-\infty}^{\infty} \frac{J_{|m|}(ka) - \lambda J_{|m|}'(ka)}{H_{|m|}^{(1)}(ka) - \lambda H_{|m|}^{(1)'}(ka)} e^{im\theta} \right|^2. \quad (150)$$

On the other hand, again by (46), the scattering amplitude associated with  $H^\lambda$ , with non-zero magnetic field, that is,  $0 < \alpha < 1$ , is given by

$$f_\alpha^\lambda(k, \theta) = \frac{1}{(2\pi i k)^{1/2}} \sum_{m=-\infty}^{\infty} \left( e^{2i\delta_m^\lambda(k, \alpha)} - 1 \right) e^{im\theta}, \quad (151)$$

and since

$$e^{2i\delta_m^\lambda(k, \alpha)} = -e^{2i\Delta_m(\alpha)} \left[ \frac{H_{|m+\alpha|}^{(2)}(ka) - \lambda H_{|m+\alpha|}^{(2)'}(ka)}{H_{|m+\alpha|}^{(1)}(ka) - \lambda H_{|m+\alpha|}^{(1)'}(ka)} \right], \quad (152)$$

we obtain

$$\begin{aligned} f_\alpha^\lambda(k, \theta) &= \frac{1}{(2\pi i k)^{1/2}} \\ &\times \sum_{m=-\infty}^{\infty} \left( -e^{2i\Delta_m(\alpha)} \left[ \frac{H_{|m+\alpha|}^{(2)}(ka) - \lambda H_{|m+\alpha|}^{(2)'}(ka)}{H_{|m+\alpha|}^{(1)}(ka) - \lambda H_{|m+\alpha|}^{(1)'}(ka)} \right] - 1 \right) e^{im\theta}. \end{aligned} \quad (153)$$

Now, let  $n \in \mathbb{Z}$  be fixed and change variable  $m' = m + n$  in the summation index. Then  $m = m' - n$  and since  $\Delta_m(\alpha) = (\pi/2)(|m| - |m + \alpha|)$ , we obtain

$$\begin{aligned} f_\alpha^\lambda(k, \theta) &= \frac{e^{-in\theta}}{(2\pi i k)^{1/2}} \sum_{m'=-\infty}^{\infty} \left( -e^{2i\delta_{m'}(\alpha-n)} (-1)^n \right. \\ &\quad \times \left. \left[ \frac{H_{|m'+\alpha-n|}^{(2)}(ka) - \lambda H_{|m'+\alpha-n|}^{(2)'}(ka)}{H_{|m'+\alpha-n|}^{(1)}(ka) - \lambda H_{|m'+\alpha-n|}^{(1)'}(ka)} \right] - 1 \right) e^{im'\theta}, \end{aligned} \quad (154)$$

which can be written as

$$f_\alpha^\lambda(k, \theta) = (-1)^n e^{-in\theta} f_{\alpha-n}^\lambda(k, \theta) + (2\pi/ik)^{1/2} [(-1)^n - 1] \delta(\theta), \quad n \in \mathbb{Z}. \quad (155)$$

Thus, the differential cross section for the Robin self-adjoint extension of the initial AB hamiltonian is given by ( $\theta \neq 0$ )

$$\left( \frac{d\sigma}{d\theta} \right)_\alpha^\lambda(k, \theta) = \frac{1}{2\pi k} \times \left| \sum_{m=-\infty}^{\infty} \left( e^{2i\Delta_m(\alpha)} \left[ \frac{H_{|m+\alpha|}^{(2)}(ka) - \lambda H_{|m+\alpha|}^{(2)'}(ka)}{H_{|m+\alpha|}^{(1)}(ka) - \lambda H_{|m+\alpha|}^{(1)'}(ka)} \right] + 1 \right) e^{im\theta} \right|^2, \quad (156)$$

which is periodic in  $\alpha$  with period 1. This is a justification for the restriction  $0 \leq \alpha < 1$ . It is convenient to write

$$f_\alpha^\lambda(k, \theta) = f_\alpha(k, \theta) + f_{r,\lambda}(k, \theta), \quad (157)$$

where  $f_\alpha$  is the scattering amplitude of the case of radius zero  $a = 0$  with Dirichlet condition at the origin, which was discussed above,

$$f_\alpha(k, \theta) = (2\pi ik)^{-1/2} \sum_{m=-\infty}^{\infty} \left( e^{2i\Delta_m(\alpha)} - 1 \right) e^{im\theta}, \quad (158)$$

and with  $f_{r,\lambda}$  given by

$$f_{r,\lambda}(k, \theta) = - \left( \frac{2}{\pi ik} \right)^{1/2} \sum_{m=-\infty}^{\infty} e^{2i\Delta_m(\alpha)} \frac{J_{|m+\alpha|}(ka) - \lambda J'_{|m+\alpha|}(ka)}{H_{|m+\alpha|}^{(1)}(ka) - \lambda H_{|m+\alpha|}^{(1)'}(ka)} e^{im\theta}. \quad (159)$$

By the same argument presented above, the series for  $f_{r,\lambda}$  is convergent, and  $f_\alpha(k, \theta)$  is given by (145).

Therefore, the differential cross section for the Robin extension with parameter  $\lambda$ , for  $k \neq 0$  and  $\theta \neq 0$ , is given by

$$\left( \frac{d\sigma}{d\theta} \right)_\alpha^\lambda(k, \theta) = \left| \frac{\sin(\pi\alpha)}{(2\pi ik)^{1/2}} \frac{e^{-i\theta/2}}{\sin(\theta/2)} - \left( \frac{2}{\pi ik} \right)^{1/2} \sum_{m=-\infty}^{\infty} e^{2i\Delta_m(\alpha)} \frac{J_{|m+\alpha|}(ka) - \lambda J'_{|m+\alpha|}(ka)}{H_{|m+\alpha|}^{(1)}(ka) - \lambda H_{|m+\alpha|}^{(1)'}(ka)} e^{im\theta} \right|^2. \quad (160)$$

Again,  $\lambda = 0$  corresponds to the Dirichlet case, whereas  $\lambda = \infty$  to the Neumann boundary condition.

### 3.4 Scattering comparison

In this section we present some figures and comments to illustrate and compare the scattering results obtained in the previous subsections. In the figures, we have fixed the value of the solenoid radius to  $a = 1$ . Due to the symmetry of the differential cross sections as function of  $\theta$ , the corresponding plots are presented only for  $0 < \theta \leq \pi$ . Recall that the scattering in case  $\alpha = 0$  is simply due to the solenoid of non-zero radius, and one notices the Aharonov-Bohm effect by comparing this case with the scattering for different values of  $\alpha$  (in particular for non-integer  $\alpha$ ).

In the following, we collect the expressions for the scattering operators for Dirichlet, Neumann and Robin extensions, respectively,

$$\begin{aligned} S_{\alpha,m}^{\mathcal{D}} &= \frac{\cos \beta [N_{|m+\alpha|}(ka)^2 - J_{|m+\alpha|}(ka)^2] - 2 \sin \beta J_{|m+\alpha|}(ka) N_{|m+\alpha|}(ka)}{N_{|m+\alpha|}(ka)^2 + J_{|m+\alpha|}(ka)^2} \\ &+ i \frac{\sin \beta [N_{|m+\alpha|}(ka)^2 - J_{|m+\alpha|}(ka)^2] + 2 \cos \beta J_{|m+\alpha|}(ka) N_{|m+\alpha|}(ka)}{N_{|m+\alpha|}(ka)^2 + J_{|m+\alpha|}(ka)^2}, \end{aligned} \quad (161)$$

$$\begin{aligned} S_{\alpha,m}^{\mathcal{N}} &= \frac{\cos \beta [N'_{|m+\alpha|}(ka)^2 - J'_{|m+\alpha|}(ka)^2] - 2 \sin \beta J'_{|m+\alpha|}(ka) N'_{|m+\alpha|}(ka)}{N'_{|m+\alpha|}(ka)^2 + J'_{|m+\alpha|}(ka)^2} \\ &+ i \frac{\sin \beta [N'_{|m+\alpha|}(ka)^2 - J'_{|m+\alpha|}(ka)^2] + 2 \cos \beta J'_{|m+\alpha|}(ka) N'_{|m+\alpha|}(ka)}{N'_{|m+\alpha|}(ka)^2 + J'_{|m+\alpha|}(ka)^2}, \end{aligned} \quad (162)$$

and

$$\begin{aligned} S_{\alpha,m}^{\lambda} &= \frac{\cos \beta \left[ \left( N_{|m+\alpha|}(ka) - \lambda N'_{|m+\alpha|}(ka) \right)^2 - \left( J_{|m+\alpha|}(ka) - \lambda J'_{|m+\alpha|}(ka) \right)^2 \right]}{\left( N_{|m+\alpha|}(ka) - \lambda N'_{|m+\alpha|}(ka) \right)^2 + \left( J_{|m+\alpha|}(ka) - \lambda J'_{|m+\alpha|}(ka) \right)^2} \\ &- \frac{2 \sin \beta \left( J_{|m+\alpha|}(ka) - \lambda J'_{|m+\alpha|}(ka) \right) \left( N_{|m+\alpha|}(ka) - \lambda N'_{|m+\alpha|}(ka) \right)}{\left( N_{|m+\alpha|}(ka) - \lambda N'_{|m+\alpha|}(ka) \right)^2 + \left( J_{|m+\alpha|}(ka) - \lambda J'_{|m+\alpha|}(ka) \right)^2} \\ &+ i \left[ \frac{\sin \beta \left[ \left( N_{|m+\alpha|}(ka) - \lambda N'_{|m+\alpha|}(ka) \right)^2 - \left( J_{|m+\alpha|}(ka) - \lambda J'_{|m+\alpha|}(ka) \right)^2 \right]}{\left( N_{|m+\alpha|}(ka) - \lambda N'_{|m+\alpha|}(ka) \right)^2 + \left( J_{|m+\alpha|}(ka) - \lambda J'_{|m+\alpha|}(ka) \right)^2} \right. \\ &\left. + \frac{2 \cos \beta \left( J_{|m+\alpha|}(ka) - \lambda J'_{|m+\alpha|}(ka) \right) \left( N_{|m+\alpha|}(ka) - \lambda N'_{|m+\alpha|}(ka) \right)}{\left( N_{|m+\alpha|}(ka) - \lambda N'_{|m+\alpha|}(ka) \right)^2 + \left( J_{|m+\alpha|}(ka) - \lambda J'_{|m+\alpha|}(ka) \right)^2} \right], \end{aligned} \quad (163)$$

and recall that  $\beta = \pi(|m| - |m + \alpha|)$ .

Figure 1 presents the real parts of scattering operators for the above three extensions, in a case with  $\alpha \neq 0$ . Note that, for high energies, the curve of the Robin scattering operator approaches the curve of the Neumann case, and it is evident the phase difference between the Dirichlet and Neumann cases; this agrees with the theoretical results of Subsection 3.3.2. For very low energies ( $ka \rightarrow 0$ ), this figure illustrates what we have said in the last paragraph of Subsection 3.3.2 about the behaviour of the scattering operator (since  $\cos \beta = 0$ ), that is, for low energies the scattering operator is very similar in all cases we have considered. Similar results hold when no magnetic field is present, i.e.,  $\alpha = 0$ .

For each of such self-adjoint extensions, we have numerically checked (with plots) that the scattering operators, with non-zero magnetic fields (i.e., for any  $\alpha \neq 0$ ), approach the corresponding scattering operators with no magnetic field (i.e.,  $\alpha = 0$ ) for high energies (not shown). Hence, given one of those self-adjoint extensions, for high energies the scattering operator is not able to discern the presence of magnetic field inside the solenoid or not. We note that such behaviours of the scattering operators were found to be independent of the values of  $m$ ,  $0 < \alpha < 1$ , and  $\lambda > 0$ .

Now, we consider the important concept of differential cross section, in the case of cylindrical solenoids of positive radius  $a > 0$ , and for the Dirichlet, Neumann and Robin extensions. The respective expressions we have obtained are, for  $\theta \neq 0$ ,

$$\left(\frac{d\sigma}{d\theta}\right)_\alpha^{\mathcal{D}}(k, \theta) = \left| \frac{\sin(\pi\alpha)}{(2\pi ik)^{1/2}} \frac{e^{-i\theta/2}}{\sin(\theta/2)} - \left(\frac{2}{\pi ik}\right)^{1/2} \sum_{m=-\infty}^{\infty} e^{2i\Delta_m(\alpha)} \frac{J_{|m+\alpha|}(ka)}{H_{|m+\alpha|}^{(1)}(ka)} e^{im\theta} \right|^2, \quad (164)$$

$$\left(\frac{d\sigma}{d\theta}\right)_\alpha^{\mathcal{N}}(k, \theta) = \left| \frac{\sin(\pi\alpha)}{(2\pi ik)^{1/2}} \frac{e^{-i\theta/2}}{\sin(\theta/2)} - \left(\frac{2}{\pi ik}\right)^{1/2} \sum_{m=-\infty}^{\infty} e^{2i\Delta_m(\alpha)} \frac{J'_{|m+\alpha|}(ka)}{H_{|m+\alpha|}^{(1)'}(ka)} e^{im\theta} \right|^2, \quad (165)$$

and

$$\left(\frac{d\sigma}{d\theta}\right)_\alpha^{\lambda}(k, \theta) = \left| \frac{\sin(\pi\alpha)}{(2\pi ik)^{1/2}} \frac{e^{-i\theta/2}}{\sin(\theta/2)} - \left(\frac{2}{\pi ik}\right)^{1/2} \sum_{m=-\infty}^{\infty} e^{2i\Delta_m(\alpha)} \frac{J_{|m+\alpha|}(ka) - \lambda J'_{|m+\alpha|}(ka)}{H_{|m+\alpha|}^{(1)}(ka) - \lambda H_{|m+\alpha|}^{(1)'}(ka)} e^{im\theta} \right|^2. \quad (166)$$



For high energies, we have found that the differential cross section of Neumann and Robin cases are very close to Dirichlet for each given  $0 \leq \alpha < 1$ , except in a neighborhood of  $\theta = 0$  and  $\theta = 2\pi$ . Figure 2 shows those curves for  $\alpha = 1/2$ .

Figure 3 shows the differential cross section of the three extensions in terms of the “energy”  $k$ , for the case with non-zero field, represented by  $\alpha = 1/2$ , fixed angle  $\theta = \pi/2$  and  $\lambda = 1$ ; note the different behaviours for high and low energies.

For  $k \rightarrow 0$ , in the case with field ( $\alpha \neq 0$ ) and positive radius, we have found that the differential cross sections for the three cases have the same behaviour, which is approximately given by the differential cross section of the case with zero radius (146) and Dirichlet condition at the origin. See Figure 4.

For intermediate energies the differential cross sections of the extensions differ significantly, as illustrated in Figure 5; this seems interesting, since it is an explicitly distinction among different boundary conditions.

Finally, we mention that for small  $\lambda$  (for example,  $\lambda = 1/10$ ), the differential cross section for the Robin extension approaches the values obtained for the Dirichlet case, and when we choose  $\lambda$  large (for example,  $\lambda = 10$ ) the values for the Neumann extension are virtually recovered. This is certainly expected.

## 4 Conclusions

With respect to the mathematical problems related to the traditional magnetic AB setting, that is, the one associated with an infinitely long solenoid, in this work we have based our investigations on two cornerstones. First, we have considered the more realistic case of a solenoid of positive radius  $a > 0$ ; and second, we did not take for granted that the boundary conditions on the solenoid border  $\mathcal{S}$  is Dirichlet (although there are physical insight [20] and mathematical arguments that support this choice [9]).

The boundary conditions that are physically compatible with quantum mechanics are those that define self-adjoint extensions of the initial AB hamiltonian (1). We have characterized all such self-adjoint extensions whose domains are contained in the natural Sobolev space  $\mathcal{H}^2(\mathcal{S}')$ ; this was done via boundary triples, and our main contribution was the inclusion of the vector potential in the operator action, by taking into account the symmetry of the problem, and a gauge choice as well, to simplify expressions.

The important cases of Dirichlet, Neumann and Robin are among the self-adjoint extensions we have characterized via boundary triples, and the next step was to study the scattering for these self-adjoint hamiltonians; such study was based on [20], where the particular case of Dirichlet boundary condition was considered. For some parameter ranges, that is,  $0 \leq \lambda \leq \infty$ ,

we have proven that the wave operators are well defined and complete; furthermore the hamiltonian is positive and has no eigenvalues. We remark that for negative values of  $\lambda$  one can not discard the presence of eigenvalues (see, for instance, Exercise 7.3.3 in [8]), and so bounded states could emerge from the Robin boundary condition; this is an interesting possibility we think it is worth investigating.

Then we have explicitly calculated the scattering operators and subsequent scattering cross sections, and they were our natural physical quantities used to compare different self-adjoint extensions. Note that the scattering cross section is a distribution in general, but for the scattering angle  $\theta \neq 0$  it is represented by a continuous function ( $k \neq 0$ ).

For high energies, we have found that the scattering operator for the Robin case is similar to the Neumann one, but different from the Dirichlet case. On the other hand, for low energies the behaviour of the scattering operator is independent of these self-adjoint extensions. Such results hold for each fixed  $0 \leq \alpha < 1$ .

With respect to the differential cross section, for “intermediate energies” its behaviour depends significantly on the choice among the three self-adjoint extensions we have considered.

To finish, we underline that, in general, our scattering results depend on the magnetic field parameter  $\alpha$ , and this is actually a confirmation of the presence of the AB effect (when  $0 < \alpha < 1$ ) in different self-adjoint extensions!

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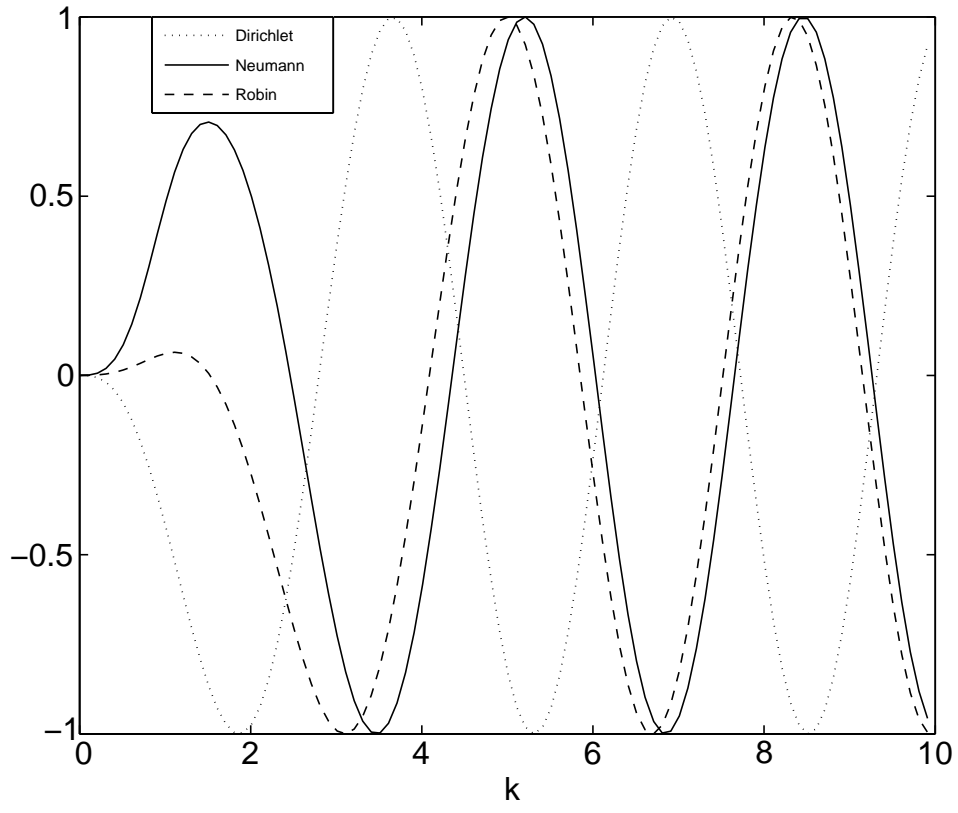


Figure 1: Real parts of the scattering operators of the three extensions with non-zero field ( $\alpha = 1/2$ ) as function of  $k$  with  $a = 1$ ,  $m = 1$  and  $\lambda = 1$ .

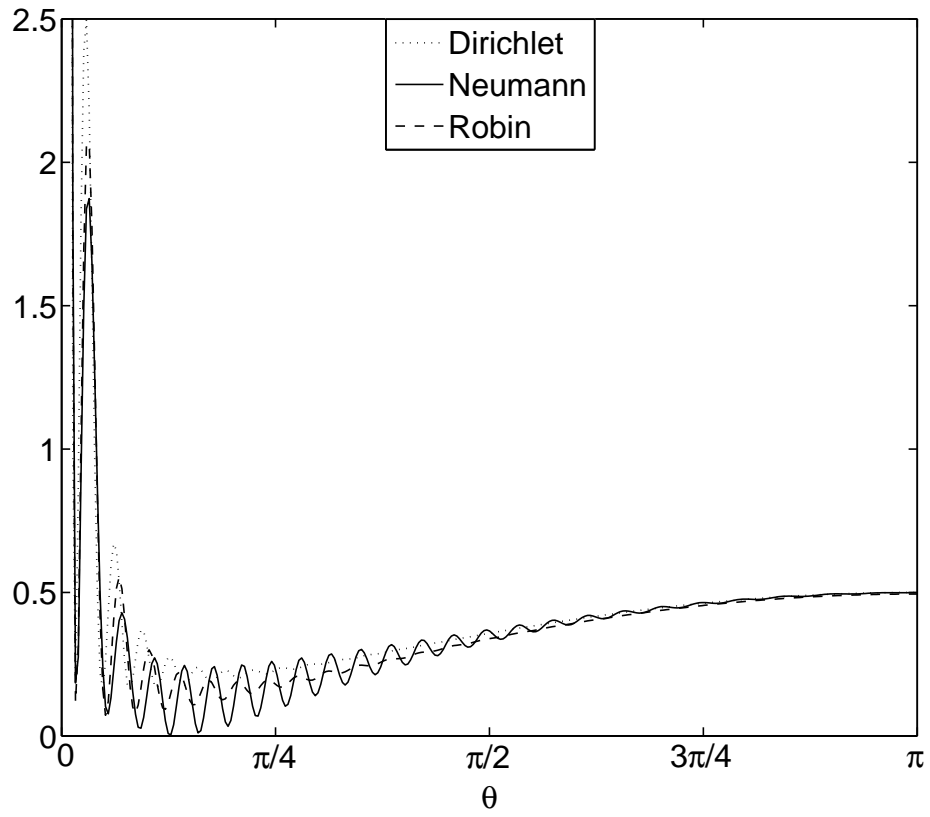


Figure 2: Differential cross section as function of  $\theta$  in the case with field ( $\alpha = 1/2$ ), with  $a = 1$ ,  $k = 30$  and  $\lambda = 1/10$ .

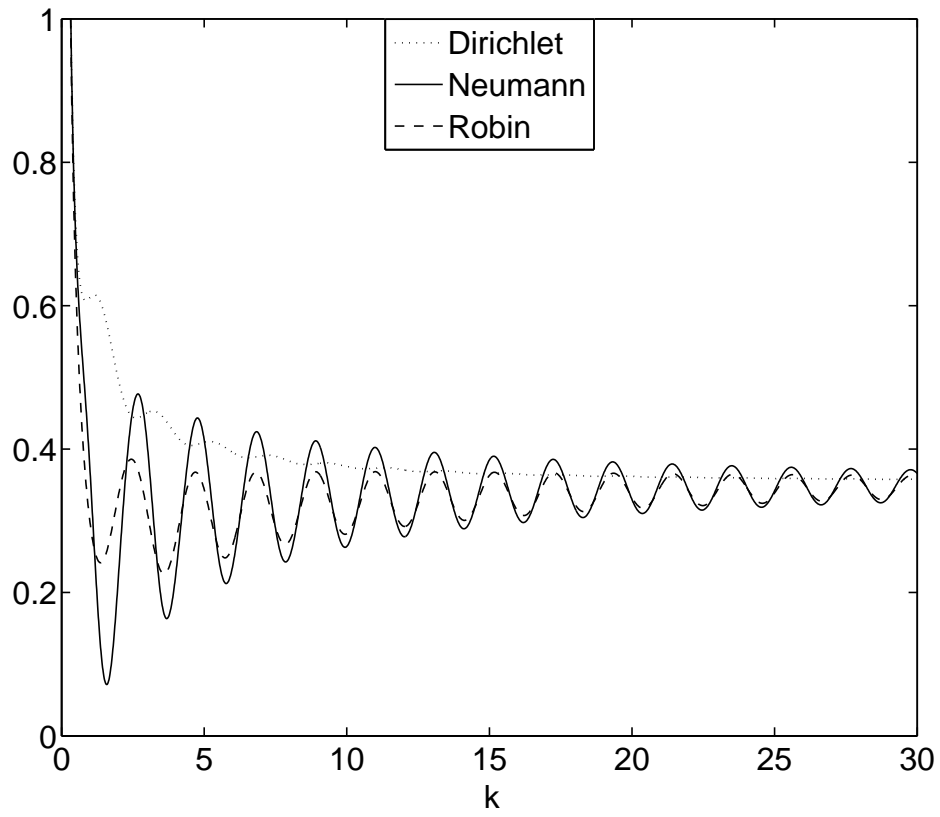


Figure 3: Differential cross section as function of  $k$  in the case with field ( $\alpha = 1/2$ ), with  $a = 1$ ,  $\theta = \pi/2$  and  $\lambda = 1$ .

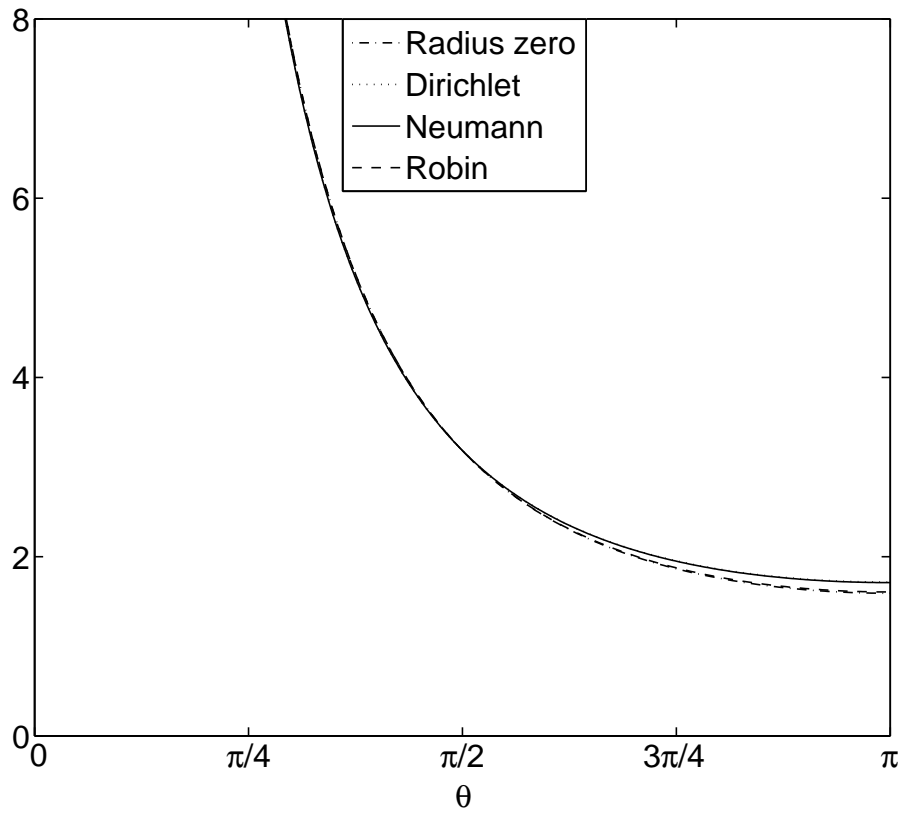


Figure 4: Differential cross section as function of  $\theta$  in the case with field ( $\alpha = 1/2$ ), with  $a = 1$ ,  $k = 1/10$  and  $\lambda = 1$ .

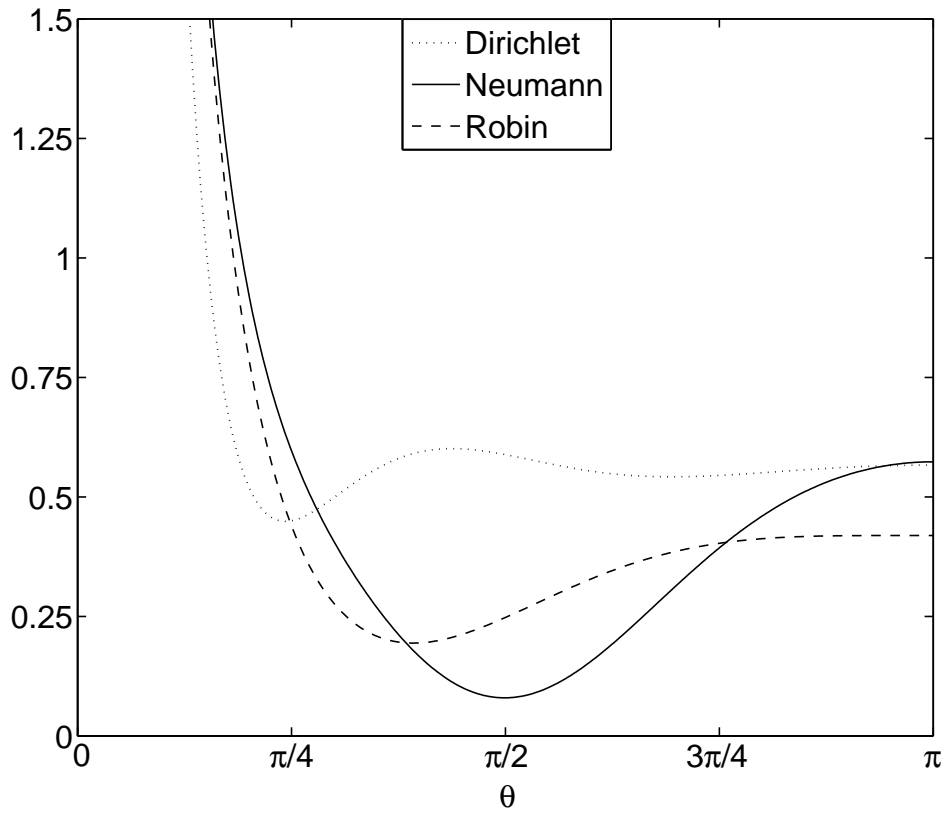


Figure 5: Differential cross section as function of  $\theta$  in the case with field ( $\alpha = 1/2$ ), with  $a = 1$ ,  $k = 3/2$  and  $\lambda = 1$ .

## References

- [1] Abramowitz M and Stegun I A 1972 *Handbook of Mathematical Functions* (New York: Dover Publications)
- [2] Adami R and Teta A 1998 On the Aharonov-Bohm hamiltonian *Lett. Math. Phys.* **43** 43–54
- [3] Adams R A and Fournier J J F 2003 *Sobolev Spaces* PAM **140** (Amsterdam: Elsevier/Academic Press)
- [4] Aharonov Y and Bohm D 1959 Significance of electromagnetic potentials in the quantum theory *Phys. Rev.* **115** 485–491
- [5] Amrein W O, Jauch J M and Sinha K B 1977 *Scattering Theory in Quantum Mechanics: Physical Principles and Mathematical Methods* (Reading: Benjamin)
- [6] Brezis H 1999 *Analyse Fonctionnelle: Théorie et Applications* (Paris: Dunod)
- [7] Dąbrowski L and Šťovíček P 1998 Aharonov-Bohm effect with  $\delta$ -type interaction *J. Math. Phys.* **39** 47–62
- [8] de Oliveira C R 2008 *Intermediate Spectral Theory and Quantum Dynamics* (Basel: Birkhäuser)
- [9] de Oliveira C R and Pereira M 2008 Mathematical justification of the Aharonov-Bohm hamiltonian *J. Stat. Phys.* **133** 1175–1184
- [10] Gradshteyn L S and Ryzhik I M 1994 *Table of Integrals, Series and Products* (San Diego: Academic Press)
- [11] Grubb G 1968 A characterization of the non-local boundary value problems associated with an elliptic operator *Ann. Sc. Norm. Sup. Pisa* **22** 425–513
- [12] Grubb G 2006 Known and unknown results on elliptic boundary problems *Bull. Amer. Math. Soc.* **43** 227–230
- [13] Grubb G 2008 *Distributions and Operators* (Berlin: Springer-Verlag)
- [14] Hagen C R 1990 Aharonov-Bohm scattering amplitude *Phys. Rev. D* **41** 2015–2017
- [15] Lions J L and Magenes E 1972 *Non-Homogeneous Boundary Value Problems and Applications* Vol. I (Berlin: Springer-Verlag)
- [16] Olver F W J 1974 *Asymptotics and Special Functions* (New York: Academic Press)



- [17] Pankrashkin K and Richard S 2009 Spectral and scattering theory for the Aharonov-Bohm operators (arXiv: math-ph/0911.4715v2)
- [18] Peshkin M and Tonomura A 1989 *The Aharonov-Bohm Effect* LNP **340** (Berlin: Springer-Verlag)
- [19] Reed M and Simon B 1979 *Methods of Modern Mathematical Physics III Scattering Theory* (San Diego: Academic Press)
- [20] Ruijsenaars S N M 1983 The Aharonov-Bohm effect and scattering theory *Ann. Phys.* **146** 1–34
- [21] Yafaev D R 1992 *Mathematical Scattering Theory: General Theory* TMM **105** (Providence: AMS)